A Theory of Payments-Chain Crises

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Abstract

I introduce payment chains into a business cycle model. Consumption decisions are linked to each other in a chain of payments. Whereas some payments can be made immediately, other payments are postponed until other payments are executed. Delays in payments delay production. An unexpected contraction in some agents ability to obtain credit leads to a payments crisis. At an initial phase, a contraction in credit sends a mass of workers to a liquidity-constrained status. This delays their payment, but also slows down the payment of others, causing a cascade in delays in a chain of payments. The real effects of these delays is isomorphic to a drop in TFP. The effects of an initial credit contraction persist even after the credit limits are normalized. I characterize these transitional dynamics and revisit some classic policy experiments.

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1. Introduction

Those of us who lived through a credit crunch can relate to the idea of a payments-chain collapse.[1] During credit crunches, it seems that firms take longer to liquidate inventories, borrowers constantly call creditors to postpone payments, and even workers may have to wait to receive their paychecks from cash-stripped employers. All in all, there's a sensation that the chain of payments has lengthened and the speed of economic transactions slowed down. In ways that are yet to be understood, this disruptions seem to carry effects on economic performance. Most strikingly, the slow down in transactions seems to persist even several periods after credit conditions have eased. What at initial dates seems to be a driven by a credit crunch, eventually evolves into a crisis of payments.

This paper rationalizes the idea of disruptions in the payments chains in a business cycle model. The theory has three distinguishing features: First, the speed of economic transactions is a function of the length of the payments chain. Second, the length of the payments chain is a function of credit market conditions. Third, a credit crunch slows down the speed of transactions, but their speed may remain low even after credit conditions are eased. These features are embedded into a business cycle model where I study how the consumption-savings decisions of agents endogenously determines the length of the payments chain in a give period. The theory showcases how payments chain disruptions manifest as a pecuniary externality and opens the door for some novel policy implications.

Payment Chains. The staring point of the theory is a sub-model where expenditure decisions and production decisions are linked through a pay-

[1] In my case, at an early age of 16, I lived through the credit crunch provoked in Peru by the1998 Russian financial crisis. I was working part time in a retail company whose credit lines were suddenly withdrawn.

ments chain. This sub-model builds on a very natural observation: payments chains link the transactions of households. The expenditures of one household is the income of a second household. In turn, the expenditures of a second household are the income of a third household, and so on. If some households cannot spend before they receive income and, moreover, have to wait between the time they receive income and spend it, they will transact with other households with some delay. Greater delays reduce the capacity to produce output efficiently, because production cannot start until some payments are made.

Of course, credit enables borrower households to spend before earning income. Hence, the provision of credit is a determinant of the speed of transactions. When credit conditions are so strong that any household can spend before earning income and there are no delays in production. But when some when credit conditions are tight, transactions are postponed and output is lost.

To capture this idea, the starting point of the sub-model is a distribution of desired transactions. Some of these transactions are predetermined to be executed on the spot and others through chained payments. Unlike a Walrasian environment, but very much like in the spirit of money-search economy, transactions are not simultaneous (not centralized). On the contrary, nature randomly links different households through a network of payments and production: a household can only buy with another household. Transactions that are executed through the payments chain are transactions where income must proceed expenditures. By contrast, the spot transactions can be executed immediately. The model produces a distribution of linked transactions where the length of payments chain is distributed geometrically with an endogenous probability that depends on the fraction of spot transactions. The greater the amount of spot transactions, the smaller the average payments chain and the faster the speed of production. I derive a formula that links output and TFP to the fraction of spot transactions.

This sub-model showcases that the distribution of means of payments determines the speed of transactions and average productivity. The presence of linked payments leads to externalities. The nature of the externality is that when an individual agent is able to transact earlier, because he makes spot payments, a second agent will earn income earlier. By earning income earlier, the second agent will transact at an earlier date, thereby speeding up the whole network of transactions. This increases real income, and a greater and reinforcing demand for credit. A theme in the second part of the paper is that the distribution of spot and chained transactions is endogenous to household overall consumption and savings decisions. However, consumption and savings decisions do not internalize the effects of their externalities.

Payment Chains in a cycle. I embed the payments chain into a dynamic but tractable business-cycle model. In the business cycle model, some households are borrowers and some are lenders. To consume a desired amount, the household must place an amount of spot and chained transactions orders. If the household has too much debt, it will be constrained by the amount of spot transactions it can place because it will lack credit. Consumption obtained through chained transaction orders is more expensive, given the delays. The decision to borrow or lend is endogenous and depends on the real interest rate and the distribution of spot and chained transactions of other households.

I demonstrate that in a steady state, households will not accumulate debt beyond the point where their spot transactions are constrained by their credit limits. Hence, the economy is dynamically efficient. However, this is not true during a credit crunch. A credit-crunch as a situation where agents suddenly lose access to credit lines that allow them to consume via spot transactions despite their debt holdings. During a credit crunch, borrowers must consume only via chained transactions before they can spend. This delay, in turn, slows down the cash-flow of other agents and, thus, average productivity falls, leading to a *payments-chain crisis*.

A payments crisis has two phases: the initial credit crunch phase and an aftermath phase. During the initial phase, borrowers not only delay their transactions, but they roll-over their debts. When the credit-crunch is over, their debt remains high so they continue to consume with chained transactions. Thus, even-though credit conditions remain deteriorated, the payments-chain continues to be inefficient. The failure to internalize effects on the payments chain will explain why the speed of transactions is inefficiently slow, many periods after the initial financial disruptions were dissipated.

Policy Implications. I then study a Ramsey planner problem and articulate how a planner that respects the technology and takes interest-rates as given, would design its expenditure path during a crunch. The planner cuts back the consumption of borrowers, something that prompts a lower real interest than otherwise. This provokes a higher consumption path by savers, but since savers consume spot transactions, this reduces the overall length of the payments chains and improves outcomes. The planner trades-off consumption smoothing with the reduction in TFP.

In a final theme, I revisit some classic discussion on fiscal-monetary policy: I first study the benefits of a savings tax. I then highlight the beneficial effects of monetary transfers to borrowers, but show that these are more potent if they are used to spend rather than to repay debt. I finally study how government expenditures are beneficial if they are executed via spot transactions, but detrimental if the government does not pay instantaneously.

Organization. The paper proceeds as follows.

Literature Review

The paper falls in the cross-road of several branches areas of study in macro-economics. First, it is connected to the literature that studies the relevance of payements frictions. Second, it connects to the literature on aggregate demand externalities. Finally, it connects to the literature on networks.

Payments Literature.

- Freeman (1996, 1999), (Lagos et al., 2011; Lagos and Rocheteau, 2009; Li et al., 2012; Nosal and Rocheteau, 2011; Rocheteau et al., 2018; Rocheteau and Rodriguez-Lopez, 2013; Rocheteau, 2011; Rocheteau et al., 2016)
- Shares the spirit of new-Monetarist literature, in that the core problem is the lack of credit or liquidity. Yet, it doesn't really on Lagos and Wright because the distribution of wealth is critical., (Green and Zhou, 2002)(Green and Zhou, 2002; Green, 1999; Green and Zhou, 1998), (Stiglitz and Greenwald, 2003)
- Can be thought of as the typical Lagos-Wright model, but where the DM meats at random times. All payments are electronic —occur simultaneously.

Chains.

- Guerrieri and Lorenzoni (2009), Kiyotaki and Moore (1997), , La'O Jams, Guerrieri, Lorenzoni, Straub and Werning...

Demand Externalities. In the model, there is a demand externality. The unconstrained agents don't internalize that by spending more, they will speed up the "the payments-chain". Thereby accelerate the use of products. In that sense, it is similar to models that have a demand externality, most notably the new-Keynesian model, or version of real models where the zero-lower bound causes a reduction in output.

- Lorenzoni (2009), Diamond (1982)
- (Guerrieri and Lorenzoni, 2017), (Korinek and Simsek, 2016), (Eggertsson and Krugman, 2012), (DÃ¡vila and Korinek, 2018)
- The closest paper to this one is Woodford (2020).

Network papers.

- Alvarez-Barlevy, Golub-Jackson,

Japanese crisis.

- Cho?

Russian closure of payments system.

2. Payment-Chains and Productivity

This section presents a simple environment meant to capture the main idea in this paper, delays in the payments chain and its connection to production. I then adapt the environment in this section and introduce it as a block of the dynamic business cycle model that appears below.

Environment. I study a collection of consumption and production orders linked through a payments chain. Time τ runs continuously over a unit interval $[0, 1]$. All transactions in this chain occur within that interval. The economy is populated by an equal number, N, of workers and shoppers. Workers and shoppers are assigned an identity $i \in \mathcal{N} = \{1, 2, \ldots, N\}$. I describe the environment for a finite N to fix ideas, but derive results for $N \to \infty$.

I will define shopper-worker relations that, when put together into a payment chain, will shape the production in the economy. The production relation $\mathcal{P}$ is a one-to-one assignment between shoppers and workers. In this relation the worker's output can only be consumed by its shopper and the shopper can only buy goods from its worker. For reasons that will become apparent later a pair of related shopper and worker cannot have the same identity.[2]

The income relation is the union of two other relations. One is the couple relation $\mathcal{X}$ where the shopper and worker in these couples have the same identity. The shopper in this couple does not have funds at $\tau = 0$ and once the worker gets paid, at some $\tau > 0$, she can transfer the funds to the shopper.

In contrast to the production relation, in the income relation there are workers and shoppers, not coupled, called singles.[3] A single shopper shows up to the payments chain with a means of payment at $\tau = 0$. This implies that they can place a shopping order immediately at time $\tau = 0$. Thus the singles relation $\mathcal{S}$ is the identity relation over the set of singles.[4] The income relation then is the union of both relations $\mathcal{I} = \mathcal{X} \cup \mathcal{S}$ and is simply the identity relation.

For now, we take their relative populations as given. We assume that a fraction μ of the N same identity worker-shopper pairs is coupled. In the dynamic model below, this fraction is endogenous and is derived from household decisions.

[2]Formally this is a bijection $\mathcal{P} : \mathcal{N} \to \mathcal{N}$ such that if $\mathcal{P}(i) = j$ then shopper i buys from worker j and $i \neq j$.

[3]Formally, let $A \subset \mathcal{N}$ be the set of couples, the couple relation is the identity bijection on A, $\mathcal{X} : A \to A$

[4]The singles relation is the identity bijection over A^c, $\mathcal{S} : A^c \to A^c$.

Payment Chains. Notice that, whereas identity defines the couple relation, we let nature choose among the possible production relations $\mathcal{P}$ with equal probability. These two underlying relations produce a network of payment chains, which is my ultimate object of study. Consider a single shopper i related with worker j (i.e. $\mathcal{P}(i) = j$), this creates a payments link because shopper i will pay worker j for producing goods. If worker j is single, any funds it receives will remain with her and there are no more payments link. Thus, this chain of payments would have length zero. However, if j is coupled she will transfer the funds to its shopper. This creates a link between shopper i and shopper j. In turn, shopper j is related to worker k (i.e. $\mathcal{P}(j) = k$), thus, the funds received from shopper i that were transferred to shopper j will be transferred to worker k. If k happens to be single, then there are no more chained payments. As a result, this chain of payments would have had length one. However, if k happens to be coupled, she will transfer funds to her shopper, and the chain will continue. Notice that the length of the payments chain is the number of links between consecutive shoppers. Below we will see that this is equal to the number of couples in a payment chain.

Characterization. A payment chains network is a sequence of payment relations where a shopper buys from a worker and the worker, possibly, funds its couple's consumption which will generate a payment to a second worker and so on and so forth until a coupled shopper pays to a single worker and the payment chain will end and restart from the single shopper.

Armed with the relations, we formally define a payment chains network by induction.

Definition 1. *A **payment chains network** is a sequence of identity pairs such that the succesor of any (i,i) is the identity pair (j,j) corresponding to the* *worker that sells to shopper i (i.e.* $\mathcal{P}(i) = j$*). Further, for any pair* (i, i) *we can say whether it is coupled or single.*

Intuitively, the network advances because shoppers pay to workers that are immediately to their right and the payment chain continues if the worker is coupled.

Since the pairs are identity pairs we could characterize the network through a **sequence representation** by a sequence of identities $\mathcal{R} = \{i_n\}_{n\geq 0}$ where $i_{n+1} = \mathcal{P}(i_n)$. For example,

$$\mathcal{R} = \{\ldots, i, j, k, l, m, n, o, p, \ldots\}.$$

Namely, j is the worker of shopper i in the production relation, k is the worker of shopper j and so on. We can further characterize the network $\mathcal{R}$ as through a **binary representation** by a sequence of singles and couples $\mathcal{B} = \{b_n\}_{n\geq 0}$ where $b_n = s$ if the n-th pair is single and $b_n = x$ if the n-th pair is a couple. For example,

$$\mathcal{B} = \{\ldots, s, s, x, s, x, x, x, s, x, \ldots\}$$

where i and j are singles, k is coupled and so on. Notice that whenever a single succeeds a couple $(\ldots, x, s, \ldots)$, that meeting in the network represents the end of one payment chain because the couple transfers funds to a single worker who will not further transfer funds. When a single succeeds a single $(\ldots, s, s, \ldots)$, it means that a chain of payments has length zero, because no two shoppers are linked. In turn if a couple succeeds a single $(\ldots, s, x, \ldots)$ there is a payment chain of lenght one because the single shopper will pay the coupled worker whom, in turn, will transfer funds to its shopper creating a link between the single and coupled shopper. An induction argument shows that the lenght of a payment chain $\{\ldots, s, x, \ldots, x, \ldots\}$ with n consecutive couples is n.

We can further represent a network by rewritting its binary representation $\mathcal{B}$ as a **collection of payment chains**. Under this representation it only makes sense to consider networks that start with a single.[5] A **payment chain** is a finite[6] subsequence of the sequence representation of the network that starts with a single and finishes before the next single. The network in our example reduces to an ordered collection of payment chains

$$\mathcal{C} = \{\ldots, \mathcal{C}_n, \mathcal{C}_{n+1}, \mathcal{C}_{n+2} \ldots\}$$

where $\mathcal{C}_n = \{s\}$, $\mathcal{C}_{n+1} = \{s, x\}$, $\mathcal{C}_{n+2} = \{s, x, x, x\}$.

Examples. I present the following figures to illustrate the concept of a payments-chain. In this case we have $N = 8$, the production relation is characterized by the sequence $\mathcal{R} = \{1, 5, 7, 4, 6, 2, 3, 8\}$ and the income relation is characterized by $\mathcal{B} = \{s, x, s, x, x, x, s, x\}$ as explained above (e.g. shopper 1 buys from worker 5 and 1 is a single and 5 is a couple). In figure 1 we can see the income relation, in green we have the shopper-worker pairs that are coupled and the arrow from worker to shopper depicts the flow of income in this direction. In contrast, we have in blue, pairs that are single and do not exhibit a flow of income between worker and shopper.

In figure 2 I add the production relation which creates (jointly with the income relation) payment-chains. To interpret the graph, notice that the chain starts with the (single) shopper 1 who pays for production of worker 5 which is coupled and will transfer funds to shopper 5 who in turn pays for production of worker 7. However, worker 7 is single so the (blue) payment chain will end at this point. This first payment chain has length one because

[5]Because if one starts with a couple the first shopper has no means of payment and its worker couple will not produce (and will not get paid) because nobody is buying from her.

[6]I assume that a next single will always appear.

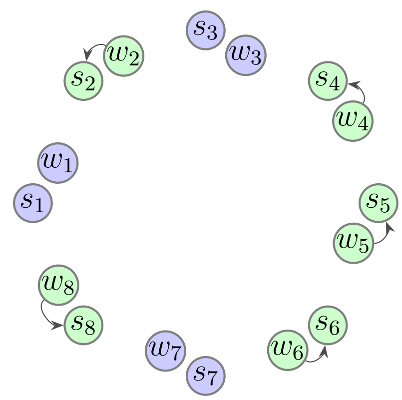

Figure 1: Income relation

it created one link between two shoppers (1 and 5), the length of the chain is also equal to the number of couples in the chain. With shopper 7 a new (orange) payment chain will start. This chain links shoppers 7, 4, 6 and 2 because 4, 6 and 2 are couples. Naturally, the length of this chain is 3. Now, the last shopper, s_2, of the second (orange) payment chain pays for production of worker 3 who stops the chain since it is single. This starts the last chain which links shopper 3 and 8.

I emphasize that in this finite case I chose to "close" the payment chain by requiring that shopper 8 buys from worker 1, this implies that the same chains will happen again and again. I will rule out this type of "cyclying" behavior in the infinite case. Namely, I will require that the payment chain advances (to new shopper-workers) and does not create a loop among the same shopper-workers.[7]

Figure 3 summarizes the information in figure 2 using the fact that the

[7]This is necessary to have randomness in chain length. Formally, this requires that nature chooses the worker i_{n+1} for shopper r_{n} among $\mathbb{N}-\left\{r_{1},\ldots,r_{n}\right\}$. Since this is still an infinite set, this drawing remains independent.

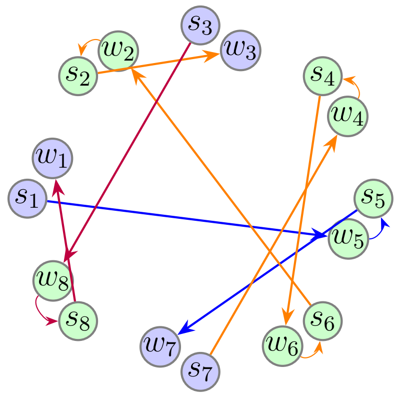

Figure 2: Payments-chain network

income relation is the identity. Again, the first (blue) chain starts with 1 and will continue up to 7 since 5 is coupled. The second chain (orange) starts at 7 and will continue up to 3 since 4, 6, 2 are coupled. The last chain starts with 3 and continuous up to 1 because 8 is coupled. This figure emphasizes the key object of study: length of chains. Abstracting from the underlying structure of the network we clearly see the number of links between shoppers for each chain.

Finally, figure 4 shows the shoppers in the left and workers in the right, the production-related payments are depicted by arrows directed to the right and the income-related payments are depicted by arrows directed to the left. For single pairs there is no flow of income (to the left) and this gives rise to new chains. In this case it is also very easy and intuitive to pick up the chain length: it is the number of colored arrow tips (for each chain) since they reflect the number of linked shoppers within a chain. We easily see that the first chain has length 1, the second length 3 and the third one length 1.

Let's now derive the distribution of payment chain lengths for $N \to \infty$. Recall that nature assigns production relations randomly. Thus, the distri-

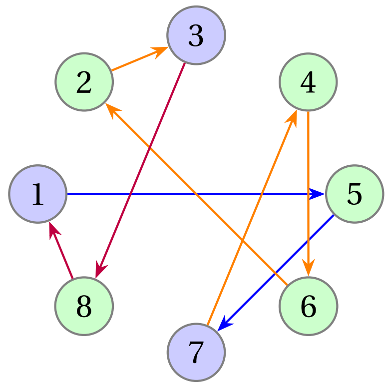

Figure 3: Summary of a payments-chain network

bution of length (n) of the payments chains follows some probability mass function depending on the proportion of couples (μ) as a parameter, namely some $G(n;\mu)$. In particular, allowing $N \to \infty$ and standing in any node of the network the probability of the next identity to be coupled or single is μ and $1-\mu$ respectively. Namely, the next type distribution is independent and identically distributed as a Bernoulli trial with probability μ. Now, a chain of payment is of length zero if the starting single (which is given) is followed by a single, this happens with probability $1-\mu$. Likewise, the chain is of length 1 if the first draw after the first single is a couple, which happens with probability μ, and the second draw is a single, which happens with probability $1-\mu$. The chain is of length two if there are two consecutive draws of couples followed by a single, and this occurs with probability $\mu^2(1-\mu)$. Proceeding by induction, we arrive at the following result:

Proposition 1. Let $n \in \{0, 1, 2, \ldots\}$ be the lenght of a payment chain, then n distributes geometrically with parameter μ, i.e., $G(n;\mu) = (1-\mu)\mu^n$ is the probability mass function of n.

We use this distribution to solve for TFP and output once we consider

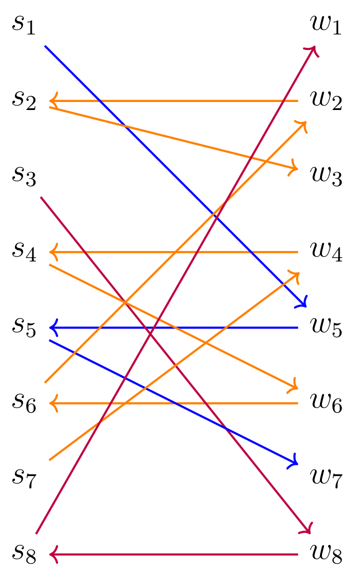

Figure 4: Payments-chain network

how payments induce delays in the production chain. In the model, what really matters is the distribution of lengths of the payment chains starting from from a single shopper up to the first single worker.

Orders, Payments, Transfers, and Production Protocols. So far we have remained silent about how and how much do shoppers pay workers and what and how much is produced. Next, I answer these questions. Shoppers uses tokens worth one unit of labor as a means of payment. As in Kiyotaki and Wright (1989), tokens are indivisible. Thus, tokens can only be used to purchase one unit of labor. At the time the shopper worker relationships are realized, shoppers agree to transfer the funds to the worker in exchange for his production, whatever its production is. All workers will carry out some amount of production. Thus, notice that there's one unit of labor employed for every shopper and since the production relation is a one-to-one map, the labor market will clear.

What is special about the environment is that the time where production takes place matters for the total production generated each period. Hence, the times at which transactions takes place will impact TFP. There are two key assumption. First, production cannot begin unless the shopper transfers tokens to the worker. Second, recall that coupled workers need to transfer funds to their coupled shopper partners. The second assumption is that the worker can transfer the tokens only after the fraction $1-\delta$ of his production is done.

Let's now discuss how production takes place. The worker has one unit of labor endowment per instant of time in the interval $[0,1]$. Since production cannot begin until she is paid, she starts production at some endogenous random time τ. Thus, she has $\sigma \equiv 1-\tau$ time available to begin produce. Then, her output will be given by: $Y_\sigma = \sigma$.

Let's see what this means for the time at which production can begin to take place for different workers in different chains. If the chain is of length 0, production can begin immediately, so the time to build is 1, so her production is 1. Next, consider a chain of lenght 1 $\{s,x\}$; the shopper pays the coupled worker and starts production immediately but she only transfer the token funds to its couple shopper at time $1-\delta$ when the time to build is δ. Next consider a chain, $\{s,x,x\}$ in which the second worker starts with δ time left to build so it produces δ and accomplishes $1-\delta$ of its production at a time that leaves δ^2 time to build for the following worker.

We see a pattern for the finishing time of transfer-required production for a chain of length n. The required production seems to end at time $1-\delta^n$ leaving δ^n time for the following worker. We see that this holds for $n=1$ and if the n-length chain finishes required production at time $1-\delta^n$ the next worker will finish its required production at time

$$1-\delta^{n}+(1-\delta)\,\delta^{n}=1-\delta^{n+1}$$

leaving δ^{n+1} time for the following worker. This proves that the time to build in a chain of lenght n are

$$\left\{1,\delta,\delta^{2},\ldots,\delta^{n}\right\}$$

where the k-th element in this sequence represents the time to build available to the $k-th$ worker in the sequence.[8] We can generalize production along many directions, but it is convenient to keep things simple this way to convey the main idea.

Statistical Properties. Our next goal is to compute the expected output for single and coupled workers given μ and the distribution of lengths I derived in Proposition 1. In turn, this allows us to compute total output as a function of μ and δ. The following Proposition provides the results.

Proposition 2. *Given μ and $N\rightarrow\infty$, the expected output of a worker in a production relation with single shoppers is 1. The expected output of a worker in a production relation with coupled shoppers is*

$$\mathcal{A}\left(\mu;\delta\right)=\frac{(1-\mu)}{\mu}\frac{\delta}{1-\delta}\ln\left(\frac{1-\delta\mu}{1-\mu}\right)<1. \tag{1}$$

Furthermore, $\mathcal{A}_{\mu}\left(\mu;\delta\right)<0$ and satisfies the following limits:

$$\lim_{\mu\rightarrow0}\mathcal{A}\left(\mu;\delta\right)=\delta \text{ and } \lim_{\mu\rightarrow1}\mathcal{A}\left(\mu;\delta\right)=0 \text{ and } \lim_{\delta\rightarrow0}\mathcal{A}\left(\mu;\delta\right)=0 \text{ and } \lim_{\delta\rightarrow1}\mathcal{A}\left(\mu;\delta\right)=1.$$

[8] Though bare in mind that in a n-length chain the n-th worker will be a single worker and it does not appear in the binary representation since it always ends with the last couple.

In turn, average output per hour is:

$$\mathcal{Y}(\mu) = (1-\mu) + \mu\mathcal{A}(\mu) < 1.$$

Furthermore, $\mathcal{Y}_{\mu}(\mu) < 0$ *and satisfies the following limits:*

$$\lim_{\mu\to 0}\mathcal{Y}(\mu;\delta) = 1 \textit{ and } \lim_{\mu\to 1}\mathcal{Y}(\mu;\delta) = 0 \textit{ and } \lim_{\delta\to 0}\mathcal{Y}(\mu;\delta) = 0 \textit{ and } \lim_{\delta\to 1}\mathcal{Y}(\mu;\delta) = 1.$$

This theorem is the key theorem in the paper. Total factor productivity, is a decreasing function of the amount of consumption that is not financed with funds directly.

- mention raw intuition: how it is constructed. Is there congestion? Not really...Ideally planner would like to distribute singles. But it cannot. Nature connects payments.
- mention structure and intuition. mention randomness in payments.
- mention limits and intuition.
- mention that $\mathcal{A}$ is also the average inverse cost of coupled consumption.
- sources of inefficiency: wages? prices? not really?

Clearly, as $\mu \to 1$, average output converges to the average output when only one firm is constrained. As $\mu \to 0$, output converges to 0, the output of two constrained firms.

Comparative Statics. Figure xxx plots different...

Discussion: Complexity. In this section, I adopt the simplest assignment protocol. I admit it is fully arbitrary, but it illustrates the potential problems that may arise with chains of payments. In general, depending on an assignment rule, there will arise a complicated combinatorial problem to be solved but not in this example.

- Discuss issues of size of transactions. Orders, etc.
- Feynman diagram footnote.

Discussion: Financial Micro-Foundations.

- Commitment before the assignment of pairs take place.
- Justify the delay δ.

3. Payment Chains in a Business-Cycle Model

We now incorporate our study of payment chains into a business cycle model. The goal of the section is to layout the environment being silent about the payment chains for now.

3.1 Environment

Timing. The timing of this model is special: Time is continuous and the horizon infinity. Financial, consumption, and labor supply decisions take place at integer dates—at $t = \{0, 1, 2, ...\}$. In turn, transactions and production take place at the time intervals between the integer dates. Those intervals correspond to the time interval studied above. The economy features is of perfect foresight, but I will study an unanticipated shock.

Demographics. The economy is populated by two classes of big-family households. One class are working-class households (workers) that are rich in human wealth, but have negative financial wealth. The other class are financially wealthy households (savers). Both households are identical except that savers do not supply labor. As in, (xxxLucas, Lucas Prescott or Shi), each household is composed of a continuum of members. Both households discount utility over time at rate β.

Transactions and Prices. To capture transactions, I organize households into a groups of single shoppers and single workers or worker-shopper couples, exactly as we did in the previous section. In particular, savers decide over an amount of consumption c_t^s which is split into a large number of spot transactions. In particular: . In turn, workers separate their h and c into ...

- explain how here consumption decision over reals is broken into constituents of smaller size (transaction size) a bit like quanta.

The unit of account is a unit of labor effort.

Saver Household. Wealthy households start each date with an amount of real deposits, D_t. Households consume and save in deposits that earn a deterministic real return R_t. The period utility is $\log(\cdot)$ over a sequence of consumption $\{c_t^s\}$. The saver's problem is given by:

Problem 1. (Saver's Problem): *Given D_0 and the path or real interest rates $\{R_{t+1}\}_{t\geq 0}$, wealthy households chose a sequence of savings $\{D_{t+1}\}_{t\geq 0}$ to maximize,*

$$\max_{\{D_{t+1}\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \log\left(C_t^s\right),$$

subject to the following budget constraint:

$$\frac{D_{t+1}}{R_{t+1}} + C_t^s = D_t,\ \forall t \geq 0.$$

The solution to this problem gives us the savers consumption path, c_t^s. Since wealthy household's must maintain positive wealth D_t to sustain positive consumption, they will always maintain positive savings. Working household's must therefore be the borrowers in this economy. Since the economy is closed, clearing on the financial markets requires:

$$D_t = B_t, \tag{2}$$

where B_t are the borrowing of workers. Thus, I proceed to the worker's problem treating B_t as worker debt. Since the saver does not supply labor, all of his consumption must be obtained from spot transactions.

Working-Class Household. Workers start each date with an amount of debt, B_t. Workers supply labor $h_t = h$ inelastically. The period utility of workers is also $\log(\cdot)$. Consumption c_t is the sum of goods obtained through spot expenditures, s_t, and chained expenditures, x_t:

$$C_t^w = S_t^w + X_t^w. \tag{3}$$

This distinction is important because whereas s_t has a unit price, the chained consumption x_t has an implicit price q_t that depends on the distribution of spot and chained payments.

Borrowings B_t is limited by the natural debt limit, $\bar{B}$, in order to prevent Ponzi schemes. This debt limit will never bind because it would mean infinite negative utility to the worker. To execute spot transactions, s_t, the worker must borrow within the period (at not interest) if it wants to execute spot transactions. I assume intra-period debt carries no interest. However, intra-period debt is limited. Namely, in addition to natural debt limit that applies to *inter-period* debt, I introduce a different constraint, a *borrowing* *limit,* $\tilde{B}$. The borrowing limit $\tilde{B}$ caps the amount of *intra-period* borrowing. Namely, spot transactions are capped as follows:

$$S_t^w \leq \max\left\{\tilde{B} - B_t, 0\right\}. \tag{4}$$

Of course, the worker can execute chained transactions in which case he does not have to borrow intra-period. However, chained consumption is costly because $q_t \geq 1$.

- add further motivation here.
- accounting goes here.

Problem 2. (Workers's Problem): *Given* B_0 *and the path or real interest rates* $\{R_{t+1}\}_{t\geq 0}$*, the worker chooses a sequence spot expenditures* S_t *and chained expenditures* X_t *to maximize:*

$$\max_{\{s_t, x_t\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \log(C_t),$$

subject to the flow budget constraint:

$$B_t + S_t^w + q_t X_t^w = \frac{B_{t+1}}{R_{t+1}} + 1, \ \forall t \geq \textit{0},$$

the definition of total consumption (3)*, the constraint on spot transactions (4), and the natural debt limit,* $B_t \leq \bar{B}$.

Equilibrium. Given the expenditure choices of both households, $\{X_t^w, s_t^w, C_t^s\}$, the ratio of chained expenditures at t is:

$$\mu_t = \frac{X_t^w}{S_t^s + S_t^w + X_t^w}.$$

This ratio define total production, TFP, and the effective cost of consumption obtained with chained expenditures:

$$Y_t = \mathcal{Y}\left(\mu_t\right), \ A_t = \mathcal{A}\left(\mu_t\right), \text{ and } q_t = \mathcal{A}\left(\mu_t\right)^{-1}. \tag{5}$$

In the economy, the goods market clearing condition is thus:

$$C_t^s + S_t^w + X_t^w = \mathcal{Y}\left(\mu_t\right) \cdot h, \tag{6}$$

and furthermore, from here we derive the following expenditure identity:

$$C_t^s + S_t^w + q_t X_t^w = h.$$

I define an equilibrium as follows.

Definition 2. *An equilibrium is a sequence of asset positions and expenditures* $\{B_t, D_t, C_t^s, S_t^w, X_t^w\}$ *together with a sequence of real rates and implicit prices* $\{R_t, q_t\}_{t\geq 0}$ *such that:*

1. *Given* $\{R_t, q_t\}_{t\geq 0}$, $\{B_t, S_t^w, X_t^w\}$ *solves the worker's problem and* $\{D_t, C_t^s\}$ *solves the saver households problem.*

2. *The asset market and goods market clears: (2) and (6).*

3. *The price* q_t *is consistent with the ratio of chained-transactions (5).*

3.2 Characterization

Solution to household problems. The wealthy household's problem is standard and its solution is recognized immediately. I summarize it in the following proposition:

Proposition 3. *Let D_0 be given. The solution to the wealthy household's problem, $\{D_{t+1}\}_{t\geq 0}$, is given by:*

$$\frac{D_{t+1}}{R_{t+1}} = \beta D_t, \textit{and} C_t^s = (1-\beta)\, D_t \; \forall t \geq \textit{0}. \tag{7}$$

We know that the worker can never consume above his labor income—combine $c^s > 0$ with the goods clearing condition.

The worker's problem is more complicated because consumption depend the fraction of spot transactions. Since spot transactions are always cheaper than chained transactions, we know that for any level of c_t^w induces the following spot and chained expenditures:

$$S_t = \min\left\{\max\left\{\tilde{B} - B_t, 0\right\}, C_t^w\right\} \text{ and } X_t = C_t^w - \min\left\{\max\left\{\tilde{B} - B_t, 0\right\}, C_t^w\right\}. \tag{8}$$

These optimal expenditures follow immediately from expenditure minimization given c_t^w. With these rules, we write the worker's problem directly in terms of its consumption choice through the following recursive representation:

Problem 3. (Modified Workers's Problem): *Given B and the path or real interest rates $\{R_{t+1}\}_{t\geq 0}$, the worker chooses consumption to maximize:*

$$\begin{aligned} \mathcal{W}_t(B_t) \quad = \quad & \max_{\{c_t^w\}_{t\geq 0}} \log(C_t^w) + ... \\ & \beta \mathcal{W}_t \underbrace{\left(R_{t+1}\left(B_t + q_t C_t^w + (q-1)\cdot \min\left\{\max\left\{\tilde{B} - B_t, 0\right\}, C_t^w\right\}\right)\right)}_{B_{t+1}}. \end{aligned}$$

The representation is obtained by replacing the optimal expenditure rules, (8) into the law of motion for debt that follows the budget constraint (xxx). This representation allows us to obtain the following generalized Euler equation.

Proposition 4. (Workers's First-Order Condition): *Consider a convergent* $\{R_t, q_t\}$ *sequence. The solution to the worker's problem,* $\{B_{t+1}\}_{t\geq 0}$ *satisfies the following first order condition:*

$$\frac{C^w_{t+1}}{C^w_t} \equiv \underbrace{\frac{1+(q_t-1)\,\mathbb{I}_{[X_t>0]}}{1+(q_{t+1}-1)\,\mathbb{I}_{[S_{t+1}=0]}}}_{\text{marginal appreciation}} \beta R_{t+1}. \tag{9}$$

The proof is not immediate from the Envelope Theorem because the budget constraint features a kink. To proof this result, I use a relaxation method: I assume c^w_t is chosen with some noise, ε, and take the noise to zero. Let's provide the intuition behind the Euler equation.

- Intuition goes here. Partition into three regions...explain that $s_{t+1} = 0$ is relevant during a credit crunch.

- Intuition: save 1 unit of chained expenditures to reduce borrowing limit and allow one unit of spot expenditures.

Steady states. Next, I characterize the set of possible steady states. In a steady state, the price q and the real rate R are fixed. I drop time subscripts and use ss to denote a steady state value. In principle, a steady state could feature chained expenditures, if $X^w_{ss} > 0$ and $q_{ss} < 1$. However, this turns out to be impossible, as the following Proposition demonstrates.

Proposition 5. *If intra-period borrowing is feasible* $\tilde{B}_{ss} > 0$*. Then, a steady state can only feature spot expenditures,* $C^w_{ss} = S^w_{ss}$ *and* $q_{ss} = 1$*. Moreover, the economy is in steady state at* t *if and only if the current debt level* B_t *satisfies:*

$$B_t \leq B^\star = \frac{1}{\beta}\left(\overline{B} - 1\right). \tag{10}$$

The proposition showcases as long as some intra-period debt borrowing is allowed, all steady-states are non-disrupted. For that to occur, condition (10) must hold. Let's provide some intuition on why this is the case. At steady state, from the saver's solution described in Proposition 3, we know that the economy can be in steady state only if $\beta R^b = 1$. Coupled with the worker's solution, 4 the worker's consumption can be in steady state if either $X > 0$ and $S = 0$ or $X = 0$ and $s > 0$. The former case can only occur if $\tilde{B}_{ss} = 0$. In turn, the latter case can only occur if the debt level of the worker is not high enough that it violated his borrowing debt limit, and this occurs when $B_t \leq B^{\star}$.

Now consider where the economy is at steady state, but a single individual worker has debt above $B^{\star}$. That worker will delever at a rate consistent with (4). As he delvers, his consumption increases up to the point where $B_t \geq B^{\star}$. If all workers are above this level, the economy transition to steady state, and this has effects on the real interest and the implicit price q.

An implication of this result, is that if there are any disruptions in the economy, these have to do with temporary low exogenous borrowing limits or temporarily high endogenous debt levels.

4. Credit-Crunch Dynamics

The previous section described that all steady states are non-disrupted. However, in this section, we describe the transitions toward a steady state and argue that they are inefficient. This the reason is that because cash-stripped households do not internalize that by shopping without cash, they lengthen the payments chains. They don't internalize that by saving a little, the following period they could all gain a marginal benefit. To do that, we first solve for the model's dynamics.

Credit crunch. We now consider a sudden unexpected decline in the borrowing limit $\tilde{B}_t$. We let the limit fall to xxx and then revert back to steady state after xxx periods. The reversion speed can be geometric, or modeled as a one time jump. To simplify the algebra, we assume it stays at 0 for T periods and then jumps back to steady state.

- [we can also do it with Poisson probability...]

Equilibrium rates and expenditures: analytic solution.

- if all workers coordinate to save, or one does...all agents abandon the condition.

We now consider a situation where the economy starts with a given level of debt, B_0, and there's s deterministic path of $\tilde{B}$.

Proposition 6. (Equilibrium Rates and Expenditures):

- *The fixed point operators go here...R equation and μ equation.*

Transitions toward steady state. [Here I describe the situation of a crunch starting from a spot steady state which terminates at a spot steady state.]

Proposition 7. (Equilibrium Rates and Expenditures):
Consider a credit crunch. If the crunch lasts for at least T^{xxx}. The economy remains inefficient for T^{xxx} additional periods.

Discussion: Borrowing vs Debt Limit. The distinction between borrowing and debt limits has technical and economic motivation: The technical motivation is that the borrowing limit allows us to study an unexpected credit crunch. Although an unexpected jump in the debt limit is not well-defined mathematically, an unexpected jump in the borrowing limit is.[9] In turn, the economic motivation is that if a bank wants to cut back on credit, it may be convenient to tighten the borrowing limit, but not necessarily to force households to repay debt principal immediately.[10]

5. Policy Discussion

Inefficiency.

- In section xxx we discussed the nature of inefficiency. As chains include more chained transaction, chains get longer. But that's a mechanical explanation. More interesting to think of debt decisions
- Desire to consume too much by poor. Debt increases, inequality increases. Hurts them. Deleveraging is too slow. Rate is too low.
- In turn, desire to consume too little by rich, currently. All in all this means the rate is too high. You want to reduce rates to discourage rich from savings. But also, inefficiency from the side of wealthy: they consume too little, attracted by high rates.
- Seems that a tax on savings is a good idea...crazy, but a good idea.
- Issue with workers is that it is too draconian. you want them to smooth consumption by less.

[9] With an unexpected change in the debt limit, there would be a positive mass of households violating their debt limits. This does not apply to the borrowing limit $\tilde{s}_t$. An alternative approach is to study a gradual shock to debt limits as in **?**.

[10] When a bank extends a loan, it increases its liabilities. This is not true about a loan rollover. During crises, banks may want to roll over debt, although they are unwilling to extend loans because the latter consumes regulatory capital. In addition, if loan repayment is suddenly forced, it can trigger default which may lead to costly underwritings.

A Ramsey Problem.

- If we endow the Ramsey planner with transfers, then it is obvious it can circumvent any credit constraints.
- If we endow the planner with only a credit tax and a transfer to workers, then consumption rule of savers is undistorted, and the interest rate absorbs the effect. Interest rate changes are akin to a transfer to savers.
- Therefore, the cleanest exercise is to introduce a time-varying uniform credit tax together with consumption taxes.
- Let B_{ss} be a steady-state level of debt and θ a Pareto weight associated with that level of debt.
- We consider a sequence of credit taxes $\{\tau_t^k\}$ and consumption taxes $\{\tau_t^c\}$ such that the Planner maximizes:

Problem 4. (Ramsey Problem):*Taking $B_0 = B_{ss}$ as given and a sequence of borrowing limits $\left\{\tilde{B}_t\right\}$, the Ramsey Planner maximizes:*

$$\max_{\{\tau_t^k, \tau_t^c\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \left[(1-\theta)\log\left(C_t^s\right) + \theta \log\left(C_t^w\right)\right],$$

subject to the saver's budget constraint and optimality conditions:

$$\frac{D_{t+1}}{\left(1-\tau_t^k\right) R_{t+1}} + \left(1+\tau_t^c\right) C_t^s = D_t, \; \forall t \geq 0$$

$$\frac{C_{t+1}^s}{C_t^s} = \beta \left[\frac{1+\tau_t^c}{1+\tau_{t+1}^c}\right] \left(1-\tau_t^k\right) R_{t+1}, \; \forall t \geq 0$$

and the workers's constraints and optimality conditions:

$$B_t + S_t^w + q_t X_t^w = \frac{B_{t+1}}{R_{t+1}} + 1, \; \forall t \geq 0$$

$$\frac{C^w_{t+1}}{C^w_t} \equiv \beta \left[\frac{1+\tau^c_t}{1+\tau^c_{t+1}}\right]\left[\frac{1+(q_t-1)\,\mathbb{I}_{[X_t>0]}}{1+(q_{t+1}-1)\,\mathbb{I}_{[S_{t+1}=0]}}\right] R_{t+1},\ \forall t \geq 0$$

$$C^w_t = S^w_t + X^w_t$$

$$S_t = \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C^w_t\right\} \text{ and } X_t = C^w_t - \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C^w_t\right\},\ \forall t \geq 0$$

and clearing in the asset market:

$$B_{t+1} = D_{t+1},\ \forall t \geq 0$$

and respecting the payments constraints:

$$\mu_t = \frac{X^w_t}{C^s_t + S^w_t + X^w_t}.$$

and the implicit cost of chained consumption $q_t = \mathcal{A}(\mu_t)^{-1}$ *and the budget balance constraint:*

$$\tau^k_t R_{t+1} B_{t+1} + \tau^c_t = 0,\ \forall t \geq 0.$$

The Ramsey planner distorts the economy with credit and consumption taxes, in order to avoid the externality. The planner takes into account the optimality conditions of the agent behavior, their constraints and their market clearing conditions. It chooses taxes subject to a budget balance condition. Naturally, there's no role for expenditures. We now turn to a primal planner problem, one where the planner can chose directly consumption of agents and choses an accounting variable, B_t, that determines dynamic constraints.

Now the problem that respects the condition.

Problem 5. (Primal Unconstrained Ramsey Problem):

Taking $B_0 = B_{ss}$ *and the time zero borrowing limit* $\left\{\tilde{B}_t\right\}_{t\geq}$ *as given, the* *primal Ramsey Planner maximizes:*

$$\max_{\{C_t^s, X_t^w, B_t\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \left[(1-\theta)\log\left(C_t^s\right) + \theta \log\left(C_t^w\right)\right],$$

subject to the resource constraint:

$$1 = C_t^s + S_t + \mathcal{A}\left(\mu_t\right)^{-1} X_t, \; \forall t \geq \boldsymbol{0}$$

$$\mu_t = \frac{X_t^w}{C_t^s + S_t^w + X_t^w}, \; \forall t \geq \boldsymbol{0}$$

$$S_t = \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C_t^w\right\} \textit{ and } X_t = C_t^w - \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C_t^w\right\}, \; t = 0.$$

$$C_t^s \leq B_t$$

Notice that the constraint set in the Primal Ramsey problem includes the constraints of the original problem. This is immediate since market clearing in the asset market and the budget balance, implies, by Walras's law that the resource constraint holds, and naturally, only the time zero credit limit is imposed.

- There are two types of constraints. Static constraints and dynamic constraints. Dynamics regards the choice of B_t.
- Static constraints regard the mix between C_t^s and X_t, given the level of S_t possible for our given construction.

The following lemma demonstrates that the solution to the primal problem coincides with the solution to original problem.

Lemma 1. (Implementability Conditions):

The solution to the Primal problem is the solution the the Ramsey problem with credit taxes and consumption taxes. The sequence of taxes that implements the solution is:

- The proof is obtained by noticing that there exists a path of $\{\tau_t^k, \tau_t^c\}$ that produces a path of consumption and B_t, consistent with the solution in the primal problem.

A special case of this problem occurs when it is possible to construct a path for B_t such that there's spot consumption only at $t = 0$. This special case is interesting, because it tells us the nature of the correction in consumption, in isolation of the path of B_t.

Problem 6. (Primal Unconstrained Ramsey Problem):

Taking $B_0 = B_{ss}$ as given and the time zero borrowing limit $\tilde{B}_0$ as given, the primal Ramsey Planner maximizes:

$$\max_{\{C_t^s, X_t^w\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \left[(1-\theta)\log\left(C_t^s\right) + \theta \log\left(C_t^w\right)\right],$$

subject to the resource constraint:

$$1 = C_t^s + S_t + \mathcal{A}\left(\mu_t\right)^{-1} X_t,\ \forall t \geq \boldsymbol{0}$$

$$\mu_t = \frac{X_t^w}{C_t^s + S_t^w + X_t^w},\ \forall t \geq \boldsymbol{0}$$

$$S_0 = \min\left\{\max\left\{\tilde{B}_0 - B_0, 0\right\}, C_0^w\right\} \textit{ and } X_0 = C_0^w - \min\left\{\max\left\{\tilde{B}_0 - B_0, 0\right\}, C_0^w\right\},\ t = 0.$$

We have the following Lemma which shows that the solution to the original Ramsey problem coincides with the primal problem.

Proposition 8. (Solution to the Primal Problem.):

The credit crunch in the primal problem lasts one period and the economy returns to steady-state immediately. Moreover, the planner distorts consumption at time zero where:

$$S_0 = \min\left\{\max\left\{\tilde{B}_0 - B_0, 0\right\}\right\}$$

and $\{X_0, C_0^s, \mu_0\}$ *solves:*

$$\frac{C_0^s}{S_0+X_0} = \frac{(1-\theta)}{\theta}\left(\frac{q(\mu_0) - q'(\mu_0)\mu_0^2 + q'(\mu_0)\mu_0}{1-q'(\mu_0)\mu_0^2}\right)$$

$$1 = C_0^s + S_0 + \mathcal{A}(\mu_0)^{-1} X_0$$

where:

$$\mu_0 = \frac{X_0}{C_0^s + S_0 + X_0}.$$

The proposition tells us that consumption is only biased in the first period of the crunch. After that, consumption can be carried out exclusively in a spot fashion. The intuition is that the credit tax has an influence on the rate of return on bonds, thus, it can distort wealth toward workers.

Clearly, the solution involves:

$$C_0^s > C_{ss}^s \text{ and } C_0^s < C_{ss}^w.$$

A more interesting question is whether the first-state consumption requires a distortion, even if the credit crunch episode vanishes after one period. If consumption at $t = 0$ is not distorted by consumption taxes, then, we want to evaluate whether there's insufficient consumption by thw savers. If their consumption is undistorted in a market solution, we obtain:

$$C_0^s = (1-\beta) B_0$$

and

$$C_0^w = \left(1 - \frac{1}{q(\mu)}\right) \min\left\{\max\left\{\tilde{B}_0 - B_0, 0\right\}\right\} + \frac{(1-(1-\beta) B_0)}{q(\mu)}$$

$$1 = q(\mu) X + \min\left\{\max\left\{\tilde{B}_0 - B_0, 0\right\}\right\} + (1-\beta) B_0$$

Proposition 9. (Insufficient consumption.): *Relative to the market solution, the solution to the Ramsey problem features more spot consumption.*

Credit Spreads: tilting the evolution of debt.

- [Do spreads work?] Akin to credit tax + transfer to worker only. Do we need to distort consumption.

Proposition 10. (....): *Relative to the market solution, the solution to the Ramsey problem features more spot consumption.*

Transfers: uses matters.

- discuss that transfer is not as useful unless used for spending. If used to pay debt, its not that good. Connect with Richard Koo's discussion.

[Here I explain that it matters if the transfer is used for transactions or to pay debt.]

Government Spending: Pay for stuff vs. Spending. We now consider the optimality of government expenditures in this environment. We consider to possibilities: the case where government expenditures are financed with current tax receipts or future tax receipts. We begin with the latter problem.

Problem 7. (Spot Government Expenditures): *Taking $B_0 = B_{ss}$ as given and a sequence of borrowing limits $\left\{\tilde{B}_t\right\}$, the Ramsey Planner maximizes:*

$$\max_{\left\{\tau_t^k, G_t\right\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \left[(1-\theta)\log\left(C_t^s\right) + \theta \log\left(C_t^w\right)\right],$$

subject to the saver's budget constraint and optimality conditions:

$$\frac{B_{t+1}}{\left(1-\tau_t^k\right) R_{t+1}} + C_t^s = B_t, \ \forall t \geq 0$$

$$\frac{C_{t+1}^{s}}{C_{t}^{s}}=\beta\left(1-\tau_{t}^{k}\right)R_{t+1},\ \forall t\geq 0$$

and the workers's constraints and optimality conditions:

$$B_{t}+S_{t}^{w}+q_{t}X_{t}^{w}=\frac{B_{t+1}}{R_{t+1}}+1,\ \forall t\geq 0$$

$$\frac{C_{t+1}^{w}}{C_{t}^{w}}\equiv\beta\left[\frac{1+(q_{t}-1)\,\mathbb{I}_{[X_{t}>0]}}{1+(q_{t+1}-1)\,\mathbb{I}_{[S_{t+1}=0]}}\right]R_{t+1},\ \forall t\geq 0$$

$$C_{t}^{w}=S_{t}^{w}+X_{t}^{w}$$

$$S_{t}=\min\left\{\max\left\{\tilde{B}_{t}-B_{t},0\right\},C_{t}^{w}\right\}\text{ and }X_{t}=C_{t}^{w}-\min\left\{\max\left\{\tilde{B}_{t}-B_{t},0\right\},C_{t}^{w}\right\},\ \forall t\geq 0$$

and respecting the payments constraints:

$$\mu_{t}=\frac{X_{t}^{w}}{C_{t}^{s}+S_{t}^{w}+G_{t}+X_{t}^{w}}.$$

and the implicit cost of chained consumption $q_{t}=\mathcal{A}\left(\mu_{t}\right)^{-1}$ *and the budget balance constraint:*

$$\tau_{t}^{k}R_{t+1}=G_{t},\ \forall t\geq 0.$$

We now study the case where G_t is consumed by the government in chained consumption. This requires expenditures to occur before tax collections.

Problem 8. (Spot Government Expenditures): *Taking* $B_0=B_{ss}$ *as given and a sequence of borrowing limits* $\left\{\tilde{B}_t\right\}$*, the Ramsey Planner maximizes:*

$$\max_{\left\{\tau_{t}^{k},G_{t}\right\}_{t\geq 0}}\sum_{t\geq 0}\beta^{t}\left[(1-\theta)\log\left(C_{t}^{s}\right)+\theta\log\left(C_{t}^{w}\right)\right],$$

subject to the saver's budget constraint and optimality conditions:

$$\frac{B_{t+1}}{\left(1-\tau_{t}^{k}\right)R_{t+1}}+C_{t}^{s}=B_{t},\ \forall t\geq 0$$

$$\frac{C_{t+1}^{s}}{C_{t}^{s}}=\beta\left(1-\tau_{t}^{k}\right)R_{t+1},\ \forall t\geq\mathit{0}$$

and the workers's constraints and optimality conditions:

$$B_{t}+S_{t}^{w}+q_{t}X_{t}^{w}=\frac{B_{t+1}}{R_{t+1}}+1,\ \forall t\geq\mathit{0}$$

$$\frac{C_{t+1}^{w}}{C_{t}^{w}}\equiv\beta\left[\frac{1+\left(q_{t}-1\right)\mathbb{I}_{\left[X_{t}>0\right]}}{1+\left(q_{t+1}-1\right)\mathbb{I}_{\left[S_{t+1}=0\right]}}\right]R_{t+1},\ \forall t\geq\mathit{0}$$

$$C_{t}^{w}=S_{t}^{w}+X_{t}^{w}$$

$$S_{t}=\min\left\{\max\left\{\tilde{B}_{t}-B_{t},0\right\},C_{t}^{w}\right\}\ \textit{and}\ X_{t}=C_{t}^{w}-\min\left\{\max\left\{\tilde{B}_{t}-B_{t},0\right\},C_{t}^{w}\right\},\ \forall t\geq\mathit{0}$$

and respecting the payments constraints:

$$\mu_{t}=\frac{X_{t}^{w}+G_{t}+}{C_{t}^{s}+S_{t}^{w}+G_{t}+X_{t}^{w}}.$$

and the implicit cost of chained consumption $q_{t}=\mathcal{A}\left(\mu_{t}\right)^{-1}$*and the budget balance constraint:*

$$\tau_{t}^{k}R_{t+1}=G_{t},\ \forall t\geq\mathit{0}.$$

We have the following result:

Proposition 11. (Expenditure multipliers): *Consider the economy where government expenditures are spot. Then, the expenditures increase the value of the Ramsey problem:*

$$\underbrace{\frac{\theta}{S^{w}+X^{w}}}_{\textit{marginal utility}}\underbrace{\frac{q\left(\mu\right)+q'\left(\mu\right)\mu\left(1-\mu\right)}{q'\left(\mu\right)\mu^{2}-1}}_{\textit{net goverment multiplier}}$$

In turn, when government expenditures are chained,

$$\underbrace{\frac{\theta}{S^{w}+X^{w}}}_{\textit{marginal utility}}\underbrace{\frac{q\left(\mu\right)+q'\left(\mu\right)\mu\left(1-\mu\right)}{q'\left(\mu\right)\left(1-\mu\right)\mu-1}}_{\textit{net goverment multiplier}}.$$

For $\mu \to 1$, we have that the spot expenditure multiplier is:

$$\frac{q(1)}{q'(1)-1} \cdots$$

and the chained expenditure multiplier:

$$\frac{q(1)}{-1} < 0$$

6. Retrospection: Japan's lost decade

7. Conclusion

There are many views about the nature of business-cycle fluctuations. Some view credit crunch episodes as the catalyst event of a subsequent economic depression. Views differ once it comes to explaining the aftermath of a credit crunch. Among these, the new-Keynesian has been particularly appealing: somehow societies can become disorganized and waste resources. What makes this idea particularly attractive is that, despite the *financial* nature of *financial* crises, there is a generalized sense that aftermath of a credit crunch have had the scent of coordination failures, or demand externalities.[11]

For monetarists, or any of us who believes that there is something inherently special about credit—*payments and money in general*—this description feels incomplete. Although there's a sense in which policy should fix the

[11] Furthermore, economic theory has made progress providing theoretical alternatives that don't conflict with rational expectations. There are many models now that present a formal interpretation of the coordination failures that underlie classic Keynesian economics—models based imperfect information, sunspots, or sentiments. Keynesian views are ready to claim success in being able to explain the sequel of a financial crises.

coordination failures, demand-driven theories are not explicit about payments in them; there's nothing financial! Yet, if one thinks about it, the deepest and longest recessions of the past century have all been triggered by problems within the financial system. The Great Depression, the Japanese lost decades, the Great Recession that has affected the US and Europe for almost a decade now, all had some financial catalyst. If coordination failures don't have monetary nature, they put all stories in the same bucket, in the same order of importance. Anything, a war, an disaster, any change in taxation, or fluctuation in the terms of trade can trigger a similar crisis. It's just a matter of how large these shocks were and the observation that the deepest recessions are financial, is just a coincidence.

The main contribution in this paper is to propose a payments interpretation financial crises and the failures that follow. The claim is that the core economic problem in the aftermath of a crises is not expectations, but an inefficient distribution of liquidity. This causes a delay in payments and is a form of coordination failures. It goes without saying that policy recommendations differ.

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Online Appendix[h]

Appendix (Nor for publication)

A. Table of Contents

Contents

B. Proofs

B.1 Proof of Proposition 2

I first derive the expected output received by couples. We aim to find the expected output of a worker in a production relation with coupled shoppers. This will be equal to the expected value of the per-worker production in a n-length chain, taking expectation across n's. Notice that in the following augmented n-length chain (augmented by the single worker in the $n+2$ position)

$$\left\{ s, \underbrace{x}_{1}, \underbrace{x}_{\delta}, \ldots, \underbrace{x}_{\delta^{n-1}}, \underbrace{s}_{\delta^{n}} \right\}$$

the production generated by workers in a production relation with n coupled shoppers is $\sum_{m=1}^{n} \delta^m$ because we do not consider the first worker for being related to a single shopper. So the per-worker production in a n-length chain is

$$\bar{y}_n^x = \frac{1}{n} \sum_{m=1}^{n} \delta^m = \frac{\delta}{n} \left(\frac{1-\delta^n}{1-\delta} \right).$$

Proof. Now, recall that a couple will necesarilly fall in a chain with length $n \geq 1$ so the distribution of lengths for couples is $G(n;\mu)$ conditional on $n \geq 1$ and since the first draw is a couple with probability μ. I have

$$G^x(n;\mu) = \frac{(1-\mu)\,\mu^n}{\mu}.$$

where G^x denotes the distribution of lengths for chains with couples. We aim to find the expected output of a worker in a production relation with coupled shoppers and it is given by

$$
\begin{aligned}
\mathbb{E}\left[\bar{y}^{x}\right] &= \sum_{n=1}^{\infty} \bar{y}_{n}^{x} G^{x}\left(n; \mu\right), \\
&= \sum_{n=1}^{\infty} \frac{\left(1-\mu\right)\mu^{n}}{\mu} \cdot \frac{\delta}{n}\left(\frac{1-\delta^{n}}{1-\delta}\right), \\
&= \frac{\left(1-\mu\right)}{\mu} \cdot \frac{\delta}{\left(1-\delta\right)} \cdot \sum_{n=1}^{\infty}\left(\frac{\mu^{n}}{n} - \frac{\left(\delta\mu\right)^{n}}{n}\right), \\
&= \frac{\left(1-\mu\right)}{\mu} \cdot \frac{\delta}{\left(1-\delta\right)} \cdot \ln\left(\frac{1-\delta\mu}{1-\mu}\right),
\end{aligned}
$$

Where the last equality comes from the fact that

$$
\begin{aligned}
\sum_{n=1}^{\infty} a^{n-1} = \frac{1}{1-a} \leftrightarrow \\
\sum_{n=1}^{\infty} \frac{a^{n}}{n} = \ln\left(\frac{1}{1-a}\right)
\end{aligned}
$$

for $|a| < 1$ because of the linearity of the derivative operator. $\square$

We call $\mathcal{Y}^{x}\left(\mu\right) = \mathbb{E}\left[\bar{y}^{x}\right]$.

Next, we derive expected output. The fraction of singles is $\left(1-\mu\right)$ and they produce on average 1 unit of output. The fraction of workers in couples is μ, and they produce on average $\mathcal{Y}^{x}\left(\mu\right)$. Thus, total output is:

$$
\begin{aligned}
\mathcal{Y}\left(\mu\right) &= \left(1-\mu\right) + \mu\mathcal{Y}^{x}\left(\mu\right) \\
&= \left(1-\mu\right) + \mu\frac{\left(1-\mu\right)}{\mu}\frac{\delta}{1-\delta}\ln\left(\frac{1-\delta\mu}{1-\mu}\right) \\
&= \left(1-\mu\right)\left(1 + \frac{\delta}{1-\delta}\ln\left(\frac{1-\delta\mu}{1-\mu}\right)\right).
\end{aligned}
$$

Next, we obtain the derivative and limits of $\mathcal{Y}\left(\mu\right)$.

Limits and Derivatives. *Proof ends here.*

This note has the proof of the worker's Euler equation

B.2 Proof of Proposition 5

A steady state with both spot and chained consumption at any period is not possible since $\beta R = 1$ in any steady state. As a result, it is enough to proof that an all chained consumption steady state is not possible. Let's suppose that $c_{ss}^{w} = X > 0$ for all periods is a steady state solution of the workers' problem. At steady state I assume that $R_t = R$ for all $t \geq 0$ so we have

$$\max_{\{X_t, S_t, B_{t+1}\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \log\left(c_t\right)$$
$$B_t + qX + \underbrace{S}_{=0} = \frac{B_{t+1}}{R} + 1$$

First, let's calculate a debt level that sustains the path of consumtion $\{(X, 0)\}_{t\geq 0}$. This is will make manipulations of the difference equation easy,

$$B_{ss} + qX = \frac{B_{ss}}{R} + 1$$
$$B_{ss} = \frac{R\left(1 - qX\right)}{\left(R - 1\right)}$$

This expression says that I can have a (positive) debt path (a constant one) as long as $qX < 1$ and the debt interest repayment is financed with the capitalized period savings. This is natural, in steady state, my per period consumption expenditure has to be lower than my real wage income. Let's first treat the case of $qX < 1$ and compute the debt path with a backward recursion

$$\begin{aligned} B_{t+1} &= RB_t - R(1-qX) \\ B_{t+1} &= R^{t+1}B_0 - \left(R^{t+1}-1\right) R\frac{(1-qX)}{R-1} \\ B_{t+1} - B_{ss} &= R^{t+1}(B_0 - B_{ss}) \end{aligned}$$

Case 1. If $B_0 - B_{ss} > 0$ we have forever increasing debt and this exceeds the natural debt limit at some finite time which will make impossible to consume X at that period in the future.

Case 2. If $B_0 - B_{ss} < 0$ then at some finite time $B_{t+1} = 0$ (at $R^{\tau+1}(B_0 - B_{ss}) = -B_{ss}$ and this necessarily happens at $\tau \geq 0$ because $B_0 > 0$). However, we only need a τ such that $0 < B_{\tau+1} < \tilde{B}$ (the spot borrowing limit) and this also happens at finite time since $B_0 > \tilde{B} > 0$. It happens at $\tau = \lceil j \rceil + 1$ where j is the time to close the intial gap and satisfies

$$\begin{aligned} B_{t+1} - \tilde{B} &= R^{j+1}(B_0 - B_{ss}) + B_{ss} - \tilde{B} = 0 \iff \\ \tilde{B} - B_{ss} &= R^{j+1}(B_0 - B_{ss}). \end{aligned}$$

In this case, in finite time (without the need of a deviation), the worker no longer has $(X, 0)$ as a solution because $S > 0$ will eventually become available and optimal.

Case 3. If $B_{ss} = B_0 > \tilde{B}$, the worker never changes the debt level and consuming $(X, 0)$ could be optimal. In this case, I cannot employ the argument above to show that this is not a steady state (because the feature before was that the debt level decreased due to the initial imbalance). However, we could use a deviation approach to show that there is an affordable and feasible plan, given prices $\{R\}_{t\geq 0}$ that achieves a higher lifetime utility. Suppose at time 0, the consumption is chosen $X - \varepsilon/q$ (for a fixed $\varepsilon > 0$) and later consumption is chosen X so we have the following equations for the path of debt

$$
\begin{aligned}
R + B_1 &= RB_0 + RqX - R\varepsilon \\
R + B_{t+1} &= RB_t + RqX, \quad \forall t \geq 1
\end{aligned}
$$

solving backwards we have

$$
\begin{aligned}
B_{t+1} &= R^{t+1}B_0 - \left(R^{t+1} - 1\right) B_{ss} - \varepsilon R^{t+1} \\
B_{t+1} - B_{ss} &= -\varepsilon R^{t+1} \\
\tilde{B} - B_{t+1} &= \varepsilon R^{t+1} - \left(B_{ss} - \tilde{B}\right)
\end{aligned}
$$

and we observe that for finite time we can have B_{t+1} as low as we want. Suppose τ is the some (need not be the first) time such that $B_{\tau+1} < \tilde{B}$, The steady state plan has utility in periods 0 and $\tau + 1$

$$
\log X + \beta^{\tau+1} \log X
$$

my deviation plan has utility in periods 0 and $\tau + 1$

$$
\log\left(X - \frac{\varepsilon}{q}\right) + \beta^{\tau+1} \log\left[X + \varepsilon R^{\tau+1} - \left(B_{ss} - \tilde{B}\right)\right]
$$

so the change in utility from deviating is

$$
\begin{aligned}
&\left\{\log\left(X - \frac{\varepsilon}{q}\right) - \log X\right\} + \beta^{\tau+1}\left\{\log\left[X + \varepsilon R^{\tau+1} - \left(B_{ss} - \tilde{B}\right)\right] - \log X\right\} \\
&\log\left(1 - \frac{\varepsilon}{X} + \frac{\varepsilon}{X}\left(1 - \frac{1}{q}\right)\right) + \beta^{\tau+1} \log\left[1 + \frac{\varepsilon}{X}R^{\tau+1} - \frac{\left(B_{ss} - \tilde{B}\right)}{X}\right]
\end{aligned}
$$

and by the mean value theorem (since log is continuous)

$$\log\left[1+\frac{\varepsilon}{X}R^{\tau+1}-\frac{\left(B_{ss}-\tilde{B}\right)}{X}\right] = \log\left[1+\frac{\varepsilon}{X}R^{\tau+1}\right]+\frac{1}{1+\frac{\varepsilon}{X}R^{\tau+1}-\omega_1\frac{\left(B_{ss}-\tilde{B}\right)}{X}}\left[-\frac{\left(B_{ss}-\tilde{B}\right)}{X}\right]$$

$$\log\left[1-\frac{\varepsilon}{X}+\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)\right] = \log\left[1-\frac{\varepsilon}{X}\right]+\frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\left[\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)\right]$$

with $\omega_1, \omega_2 \in (0,1)$, ω_1 depends on $\varepsilon R^{\tau+1}$ and ω_2 depends on ε only. Rearranging terms

$$\left\{\log\left[1-\frac{\varepsilon}{X}\right]+\beta^{\tau+1}\log\left[1+\frac{\varepsilon}{X}R^{\tau+1}\right]\right\}$$

$$+\left\{\underbrace{\frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}_{\text{lower price benefit}}-\beta^{\tau+1}\frac{1}{1+\frac{\varepsilon}{X}R^{\tau+1}-\omega_1\frac{\left(B_{ss}-\tilde{B}\right)}{X}}\underbrace{\frac{\left(B_0-\tilde{B}\right)}{X}}_{\text{initial gap}}\right\}$$

We know that the first term can be made arbitrarily small choosing ε. Let's work with the second term. For each $\varepsilon > 0$ the "lower price benefit" term is fixed and the denominator of the slope of the "initial gap" is bounded below

$$1+\frac{\varepsilon}{X}R^{\tau+1}-\frac{\left(B_{ss}-\tilde{B}\right)}{X} < 1+\frac{\varepsilon}{X}R^{\tau+1}-\omega_1\frac{\left(B_{ss}-\tilde{B}\right)}{X}$$

and since the LHS of this inequality can get arbitrarily large with some large

τ, the RHS too. Using this inequality I have

$$\frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)-\beta^{\tau+1}\frac{1}{1+\frac{\varepsilon}{X}R^{\tau+1}-\omega_1\frac{\left(B_{ss}-\tilde{B}\right)}{X}}\frac{\left(B_{ss}-\tilde{B}\right)}{X}$$
$$>\frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)-\beta^{\tau+1}\frac{1}{1+\frac{\varepsilon}{X}R^{\tau+1}-\frac{\left(B_{ss}-\tilde{B}\right)}{X}}\frac{\left(B_{ss}-\tilde{B}\right)}{X}$$

where it is easy to see that (for any fixed ε) the quantity

$$\lim_{\tau+1}\beta^{\tau+1}\frac{1}{1+\frac{\varepsilon}{X}R^{\tau+1}-\frac{\left(B_{ss}-\tilde{B}\right)}{X}}=0$$

goes to zero. So given ε fixed, we can find $k>0$ (sufficiently small) and τ depending on k satisfying

$$\underbrace{\frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}_{\text{lower price benefit}}-\beta^{\tau+1}\frac{1}{1+\frac{\varepsilon}{X}R^{\tau+1}-\omega_1\frac{\left(B_{ss}-\tilde{B}\right)}{X}}\underbrace{\frac{\left(B_0-\tilde{B}\right)}{X}}_{\text{initial gap}}>$$
$$\underbrace{\frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}_{A(\varepsilon)=\text{lower price benefit}}-k>0$$

So for every $\varepsilon>0$ we can define k's positive but smaller than $A(\varepsilon)$ such that $A(\varepsilon)-k>0$ and we can make this quantity as close as desired to $A(\varepsilon)$ (choosing a smaller $k>0$). Now returning to the change of utility of the deviation, this expression can be made arbitrarily close to

$$\left\{\log\left[1-\frac{\varepsilon}{X}\right]+\beta^{\tau+1}\log\left[1+\frac{\varepsilon}{X}R^{\tau+1}\right]\right\}+A(\varepsilon)$$

As a consequence, it only remains to show that the above quantity is positive for small ε (fixing τ). Using a taylor expansion

$$\begin{aligned}
&-\left(\frac{\varepsilon}{X}\right)^2-\beta^{\tau+1}\left(\frac{\varepsilon}{X}R^{\tau+1}\right)^2+\mathcal{O}\left(\|\varepsilon\|^3\right)+A\left(\varepsilon\right)\\
&>-\left(\frac{\varepsilon}{X}\right)^2\left[1+R^{\tau+1}\right]+\frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)\\
&=a\varepsilon-b\varepsilon^2
\end{aligned}$$

because the third derivative adjustment for log is positive[12] and where

$$a=\frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{1}{X}\left(1-\frac{1}{q}\right)>0,\quad b=\left[1+R^{\tau+1}\right]\frac{1}{X^2}>0$$

Finally, we know that a linear function (with positive coefficient) is always greater than a square (with positive coefficient) for $\varepsilon>0$ small (because the square stays very close to zero for ε small). Namely, $a\varepsilon-b\varepsilon^2>0$. So the deviation was profitable, which finishes the proof of this case.

A special case happens when the initial gap is zero, i.e. $B_{ss}=B_0=\tilde{B}$, but then we only need to wait one period after our deviation to increase spot consumption profitably. So a steady state solution with $qX<1$ and $B_0=B_{ss}=\tilde{B}$ is not possible.

Case 4. If $qX=1$ then $B_{ss}=0$ and only a zero initial level of debt is admissible. In this case, if $\tilde{B}>0$ then spot consumption is available and because (from euler equation) $X>0$, $S>0$ is incompatible with a steady state for the saver, we get a contradiction. So an all chained consumption steady state with $qX=1$ is not possible.

To seet that the economy is at steady state if and only if

$$B_t\leq B^\star=\frac{1}{\beta}\left(\overline{B}-1\right)$$

[12] $\left(-\frac{\varepsilon}{X}\right)^3+\beta^{\tau+1}\left(\frac{\varepsilon}{X}R^{\tau+1}\right)^3\to\left(\frac{\varepsilon}{X}\right)^3\left[R^{2\tau+2}-1\right]>0$

See that...

B.3 Proof of Propositions 3 and 4

Proof.

The proof of Proposition 3 is immediate. The proof of Proposition 4 requires more work. We begin with a perturbed version of Problem (xxx). Consider the following:

$$= \max .$$

Thus, we arrive at the recursion.

$$U^{'}\left(c\left(B\right)\right) \equiv \frac{\beta R_{t+1}}{\left(q-1\right)\mathbb{I}_{\left[c\left(B\right)>\Xi\left(\tilde{B},B\right)^{+}\right]}+1}U^{'}\left(c\left(RB-h+c\left(B\right)+\left(q-1\right)\left(c\left(B\right)-\Xi\left(\tilde{B},B\right)^{+}\right)^{+}\right)\right).$$

Proof ends here.

B.4 Proof of Corollary xxx

Proof.

Recall that

$$c^{r} = \left(1-\beta\right)B$$

and the expenditures of workers is:

$$s+qx = h-\left(1-\beta\right)B.$$

Since we know that

$$s \leq \Xi\left(\tilde{B},B\right)^{+}$$

we have that:

$$s = \min\left\{\Xi\left(\tilde{B},B\right)^{+},h-\left(1-\beta\right)B\right\}$$

$$x=\frac{h-(1-\beta)B-\min\left\{\Xi\left(\tilde{B},B\right)^{+},h-(1-\beta)B\right\}}{q}.$$

Then,

$$\begin{aligned}\mu &= \frac{x}{c^{r}+s+x}\\ &= \frac{\frac{h-(1-\beta)B-\min\left\{\Xi\left(\tilde{B},B\right)^{+},h-(1-\beta)B\right\}}{q}}{(1-\beta)B+\frac{h-(1-\beta)B-\min\left\{\Xi\left(\tilde{B},B\right)^{+},h-(1-\beta)B\right\}}{q}+\min\left\{\Xi\left(\tilde{B},B\right)^{+},h-(1-\beta)B\right\}}\\ &= \frac{1}{1+q\frac{(1-\beta)B+\min\left\{\Xi\left(\tilde{B},B\right)^{+},h-(1-\beta)B\right\}}{h-(1-\beta)B-\min\left\{\Xi\left(\tilde{B},B\right)^{+},h-(1-\beta)B\right\}}}.\end{aligned}$$

With this we replace:

$$\mu=\frac{1}{1+\frac{1}{\mathcal{A}(\mu)}\cdot\Gamma(B)},$$

where

$$\Gamma(B)=\frac{(1-\beta)B+\min\left\{\Xi\left(\tilde{B},B\right)^{+},h-(1-\beta)B\right\}}{h-\left((1-\beta)B+\min\left\{\Xi\left(\tilde{B},B\right)^{+},h-(1-\beta)B\right\}\right)}.$$

Then, using

$$A(\mu)=\frac{(1-\mu)}{\mu}\frac{\delta}{1-\delta}\ln\left(\frac{1-\delta\mu}{1-\mu}\right)$$

we obtain:

$$1-\mu=\frac{\mu}{\mathcal{A}(\mu)}\cdot\Gamma(B).$$

- Could have solution if inverse is there. Cross-fingers!

We obtain a fixed point problem in μ. We call this object $\mu(B)$. And $q(B)=q(\mu(B))$.

- It may also be possible to solve for $q(B)$ or $\mu(B)$.

Next, we sum x and s in xxx to obtain:

$$c\left(B\right)=\frac{h-\left(1-\beta\right)B}{q\left(B\right)}+\left(1-\frac{1}{q\left(B\right)}\right)\min\left\{\Xi\left(\tilde{B},B\right)^{+},h-\left(1-\beta\right)B\right\}.$$

Now in the modified euler equation:

$$U^{'}\left(c\left(B\right)\right)=\frac{\beta R_{t+1}}{\left(q\left(\mu\left(B'\right)\right)-1\right)\mathbb{I}_{\left[c(B)>\Xi\left(\tilde{B},B\right)^{+}\right]}+1}U^{'}\left(c\left(B'\right)\right).$$

Since we know $B'=\beta R_{t+1}B$ we obtain:

$$U^{'}\left(c\left(B\right)\right)=\frac{\beta R_{t+1}}{\left(q\left(\mu\left(\beta R_{t+1}B\right)\right)-1\right)\mathbb{I}_{\left[c(B)>\Xi\left(\tilde{B},B\right)^{+}\right]}+1}U^{'}\left(c\left(\beta R_{t+1}B\right)\right).$$

With this, we have one equation in one unknown, R.

- May be really simple for log
- The condition may hold everywhere, independent of t.

Proof ends here.

B.5 Proposition Condition...

Proof. Notice that the steady state is indeed a spot-transaction steady state, the price is $p=1$. Assume that the worker consumes spot transactions forever. Then, the labor first-order condition is

$$h^{\nu}=1.$$

Thus, there's a total of $h=1$ labor effort. Thus,

$$Y=1.$$

At steady state $R = 1/\beta$, otherwise the wealthy households will continue change their wealth. Then, there consumption is:

$$C^s = (1-\beta)\,B.$$

By the clearing condition in the goods market,

$$C = 1 - (1-\beta)\,B,$$

where the condition follows from the fact that all consumption is spot. Then,

$$s = 1 - (1-\beta)\,B \leq \tilde{B} - B.$$

Hence, the condition follows. Finally, we must show that the worker does not wish to accumulate debt. Notice that he is would not accumulate debt even if the constraint were not binding ever. Thus, he must be at an optimum. □

B.6 Proof of Proposition (xxx) (regarding the transition after a credit crunch)

Preliminary Observations. The economy starts from a given steady state ss^1. Thus, at the time of the shock, $t = 0$, $B_0 = B_{ss^1}$. From Proposition 3, we know that the optimal consumption

$$C_t^s = (1-\beta)\,B_t,\ \forall t.$$

Assume that $\tilde{B}_0 > 0$ and that the sequence is increasing. Then, the worker consumes at least some amount by executing spot transactions. Namely,

$$S_0^w = \min\left\{\tilde{B}_0 - B_0, C_0^w\right\}$$

and

$$X_0^w = \frac{C_0^w - \min\left\{\max\left\{\tilde{B}_0 - B_0, 0\right\}, C_0^w\right\}}{q_0}.$$

In particular, this happens at all periods:

$$S_t^w = \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C_t^w\right\}$$

and

$$X_t^w = \frac{C_t^w - \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C_t^w\right\}}{q_t}.$$

Now, we combine the household's expenditures

$$h = C_t^s + S_t^w + q_t X_t^w.$$

Thus, subbing (xxx) and (xxx) into (xxx), we obtain:

$$X_t^w = \frac{h - \left((1-\beta)\,B_t + \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C_t^w\right\}\right)}{q_t}.$$

Assume that the borrowing limit is binding such that X_t^w. Thus, we must have that:

$$X_t^w = \frac{h - \left((1-\beta)\,B_t + \max\left\{\tilde{B}_t - B_t, 0\right\}\right)}{q_t} > 0.$$

Item (i). Combining (xxx) and the definition of μ, (xxx), we obtain:

$$\begin{aligned}
\mu &= \frac{\frac{h-\left((1-\beta)B_t+\max\left\{\tilde{B}_t-B_t,0\right\}\right)}{q_t}}{(1-\beta)\,B_t + \max\left\{\tilde{B}_t - B_t, 0\right\} + \frac{h-\left((1-\beta)B_t+\max\left\{\tilde{B}_t-B_t,0\right\}\right)}{q_t}} \\
&= \frac{h - \left((1-\beta)\,B_t + \max\left\{\tilde{B}_t - B_t, 0\right\}\right)}{q_t\left((1-\beta)\,B_t + \max\left\{\tilde{B}_t - B_t, 0\right\}\right) + h - \left((1-\beta)\,B_t + \max\left\{\tilde{B}_t - B_t, 0\right\}\right)}.
\end{aligned}$$

Thus,

$$\frac{1}{\mu} = q_t \frac{\left(\left(1-\beta\right) B_t + \max\left\{\tilde{B}_t - B_t, 0\right\}\right)}{h - \left(\left(1-\beta\right) B_t + \max\left\{\tilde{B}_t - B_t, 0\right\}\right)} + 1.$$

Then, subtracting one from both sides, we obtain:

$$\frac{1-\mu}{\mu} \cdot \frac{1}{q_t} = \Lambda\left(B_t, \tilde{B}_t\right),$$

where

$$\Lambda\left(B, \tilde{B}\right) = \frac{\left(\left(1-\beta\right) B + \max\left\{\tilde{B} - B, 0\right\}\right)}{h - \left(\left(1-\beta\right) B + \max\left\{\tilde{B} - B, 0\right\}\right)}.$$

Then, observe that $q_t = 1/\mathcal{A}\left(\mu_t\right)$. Hence, we obtain:

$$\frac{1-\mu}{\mu} \cdot \frac{1}{q_t} = -\left(\mu_t\right).$$

where we applied the definition of $\mathcal{A}\left(\mu_t\right)$ in the second equality and use the following definition

$$-\left(\mu\right) \equiv \frac{1-\mu}{\mu} \mathcal{A}\left(\mu_t\right).$$

Thus, at any t where spot transaction are binding:

$$-\left(\mu\right) = \Lambda\left(B, \tilde{B}\right). \tag{11}$$

Notice that in any equilibrium, because B satisfies the natural debt limit, $h > \left(1-\beta\right) B$. Since chained expenditures are positive, it must be that $h - \left(\left(1-\beta\right) B + \max\left\{\tilde{B} - B, 0\right\}\right)$ is non-negative. Thus, $\Lambda\left(B, \tilde{B}\right) > 0$. Also, observe that $-\left(\mu\right)$ is decreasing in μ because by Proposition $\mathcal{A}\left(\mu_t\right)$, the term $\mathcal{A}$ is decreasing. Moreover, it has the limits: $\lim_{\mu \to 0} -\left(\mu\right) = \infty$ and $\lim_{\mu \to 1} -\left(\mu\right) = 0$. This means that there's a unique solution to (11) we denote by the function

$$\mu\left(B, \tilde{B}\right) \equiv \left\{\mu \middle| -\left(\mu\right) = \Lambda\left(B, \tilde{B}\right)\right\}.$$

There is no known analytic root to this problem.

However, in equilibrium

$$\mu_t = \mu\left(B_t, \tilde{B}_t\right)$$

and moreover:

$$q\left(B, \tilde{B}\right) = \frac{\mu\left(B, \tilde{B}\right)}{\left(1-\mu\left(B, \tilde{B}\right)\right)} \cdot \frac{(1-\delta)}{\delta} \cdot \ln\left(\frac{1-\mu\left(B, \tilde{B}\right)}{1-\mu\left(B, \tilde{B}\right)\delta}\right).$$

$$q_t = q\left(B_t, \tilde{B}_t\right).$$

Item (ii). Next, we combine the worker and savers first order condition.

Consider the first-order condition at a point where t is such that $X_t^w > 0$ and $S_t^w = 0$. Thus, we have that:

$$\begin{aligned}
C_{t+1}^w &= X_t^w + S_t^w \\
&= \frac{h-(1-\beta)B_t - S_t^w}{q_t} + S_t^w \\
&= \frac{h-(1-\beta)B_t + (q_t-1)S_t^w}{q_t} \\
&= \frac{h-(1-\beta)B_t + (q_t-1)\max\left\{\tilde{B}_t - B_t, 0\right\}}{q_t}.
\end{aligned}$$

Now consider the worker's first order condition [Important: must bind at t+1 also...else, must replace by future S, not, the binding one...just patch with

X=0]

$$
\begin{aligned}
q_t \beta R_t &= \frac{C_{t+1}^w}{C_t^w} \\
&= \frac{q_t}{q_{t+1}} \frac{h - (1-\beta) B_{t+1} + (q_{t+1} - 1) \max\left\{\tilde{B}_{t+1} - B_{t+1}, 0\right\}}{h - (1-\beta) B_t + (q_t - 1) \max\left\{\tilde{B}_t - B_t, 0\right\}}.
\end{aligned}
$$

Thus,

$$
\beta R_t = \frac{1}{q_{t+1}} \frac{h - (1-\beta) B_{t+1} + (q_{t+1} - 1) \max\left\{\tilde{B}_{t+1} - B_{t+1}, 0\right\}}{h - (1-\beta) B_t + (q_t - 1) \max\left\{\tilde{B}_t - B_t, 0\right\}}.
$$

Now, recall that $B_{t+1} = \beta R_t B_t$. Thus, we obtain:

$$
\beta R_t = \frac{1}{q_{t+1}} \frac{h - (1-\beta) \beta R_t B_t + (q_{t+1} - 1) \max\left\{\tilde{B}_{t+1} - \beta R_t B_t, 0\right\}}{h - (1-\beta) B_t + (q_t - 1) \max\left\{\tilde{B}_t - B_t, 0\right\}}.
$$

Hence, we obtain:

$$
\beta R = \frac{1}{q\left(\beta R B, \tilde{B}'\right)} \frac{h - (1-\beta) \beta R B + \left(q\left(\beta R B, \tilde{B}'\right) - 1\right) \max\left\{\tilde{B} - \beta R B, 0\right\}}{h - (1-\beta) B + \left(q\left(B, \tilde{B}\right) - 1\right) \max\left\{\tilde{B} - B, 0\right\}}.
$$

This gives us an implicit solution which we can solve:

$$
R\left(B, \tilde{B}, \tilde{B}'\right)
$$

and

$$
B'\left(B, \tilde{B}, \tilde{B}'\right) = \beta R\left(B, \tilde{B}, \tilde{B}'\right) B.
$$