Measuring $r^*$: A Note on Transitory Shocks

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Federal Reserve Board
The analysis and conclusions set forth are those of the authors and do not indicate concurrence by other members of the research staff or the Board of Governors.
“That’s based on our view that the neutral nominal federal funds rate...is currently quite low by historical standards. That means that the federal funds rate does not have to rise all that much to get to a neutral policy stance.”
Main Results

We incorporate fully the uncertainty on $r^*$ embedded in the data, we find:

- a more procyclical $r^*$ in the benchmark model.
- evidence of transitory shocks to $r^*$. 
Baseline model (Holston, Laubach and Williams 2016)

\[
\begin{align*}
\tilde{y}_t &= y_t - y^*_t \\
\tilde{y}_t &= a_{y,1}\tilde{y}_{t-1} + a_{y,2}\tilde{y}_{t-2} + \frac{a_r}{2} \sum_{j=1}^{2} (r_{t-j} - r^*_{t-j}) + \varepsilon_{\tilde{y},t} \\
\pi_t &= b_{\pi}\pi_{t-1} + (1 - b_{\pi})\pi_{t-2,4} + b_{\gamma}\tilde{y}_{t-1} + \varepsilon_{\pi,t} \\
y^*_t &= y^*_{t-1} + g_{t-1} + \varepsilon_{y^*,t} \\
g_t &= g_{t-1} + \varepsilon_{g,t} \\
r^*_t &= g_t + z_t \\
z_t &= z_{t-1} + \varepsilon_{z,t}
\end{align*}
\]
**Extended model**

\[
\begin{align*}
\tilde{y}_t & = y_t - y_t^* \\
\tilde{y}_t & = a_{y,1} \tilde{y}_{t-1} + a_{y,2} \tilde{y}_{t-2} + \frac{a_r}{2} \sum_{j=1}^{2} (r_{t-j} - r_{t-j}^*) + \varepsilon_{\tilde{y},t} \\
\pi_t & = b_\pi \pi_{t-1} + (1 - b_\pi) \pi_{t-2,4} + b_{\gamma} \tilde{y}_{t-1} + \varepsilon_{\pi,t} \\
y_t^* & = y_{t-1}^* + g_{t-1} + \varepsilon_{y^*,t} \\
g_t & = \mu_g + \rho_z (g_{t-1} - \mu_g) + \varepsilon_{g,t} \\
r_t^* & = g_t + z_t \\
z_t & = \rho_z z_{t-1} + \varepsilon_{z,t}
\end{align*}
\]
Model estimation: MLE

- pileup problem (Stock 1994)
- Solution proposed by Laubach and Williams 2003 based on medium unbiased estimator from Stock and Watson 1998:

\[
\lambda_g \equiv \frac{\sigma_g}{\sigma_{y^*}}, \quad \lambda_z \equiv \frac{a_r \sigma_z}{\sigma_{\tilde{y}}}. 
\]

LW Method:

1. Simplify model, estimate \( \lambda_g \).
2. Fix \( \lambda_g \) value, use alternative simplification, estimate \( \lambda_z \).
3. Fix \( \lambda_g \) and \( \lambda_z \), estimate remaining parameters.
Model estimation: MCMC


- We use standard Bayesian methods (random walk MC, FFBS).

- Flat priors.

- Imposing HLW $\lambda_g$ and $\lambda_z$, we replicate HLW.
### Priors

<table>
<thead>
<tr>
<th>Name</th>
<th>Domain</th>
<th>Density</th>
<th>Parameter 1</th>
<th>Parameter 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$\mathbb{R}$</td>
<td>Normal</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$\mathbb{R}$</td>
<td>Normal</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$a_r$</td>
<td>$\mathbb{R}^-$</td>
<td>Normal</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$b_1$</td>
<td>$[0, 1]$</td>
<td>Uniform</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$b_Y$</td>
<td>$\mathbb{R}^+$</td>
<td>Normal</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$\rho_g$</td>
<td>$\mathbb{R}$</td>
<td>Normal</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\mu_g$</td>
<td>$\mathbb{R}$</td>
<td>Normal</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$\rho_z$</td>
<td>$\mathbb{R}$</td>
<td>Normal</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>$[0, 5]$</td>
<td>Uniform</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>$[0, 5]$</td>
<td>Uniform</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>$\sigma_3$</td>
<td>$[0, 5]$</td>
<td>Uniform</td>
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<td>5</td>
</tr>
<tr>
<td>$\sigma_4$</td>
<td>$[0, 5]$</td>
<td>Uniform</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>$\sigma_5$</td>
<td>$[0, 5]$</td>
<td>Uniform</td>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>
MLE vs Bayesian: the difference in $\lambda$'s

$\lambda_g$

$\lambda_z$
We estimate the key parameters $\rho_g$, $\mu_g$ and $\rho_z$ of the extended model. We consider four alternatives:

- **Model I** $\rho_g = 1$, $\rho_z = 1$, (only different estimation technique)
- **Model II** $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$
- **Model III** $\rho_g = 1$, $\rho_z$ estimated
- **Model IV** $\rho_g$, $\mu_g$ and $\rho_z$ estimated
Models Considered

- We estimate the key parameters $\rho_g$, $\mu_g$ and $\rho_z$ of the extended model.
- We consider four alternatives:
  - Model I $\rho_g = 1$, $\rho_z = 1$, (only different estimation technique)
  - Model II $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$
  - Model III $\rho_g = 1$, $\rho_z$ estimated
  - Model IV $\rho_g$, $\mu_g$ and $\rho_z$ estimated
Model I: $\rho_g = 1, \rho_z = 1$

Smoothed $r^*$ Draws in Model I (Shaded 10th to 90th Percentile)
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$

Smoothed $r^*$ Draws in Model II
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$
Model III: $\rho_g = 1$, $\rho_z$ estimated

Smoothed $r^*$ Draws in Model III
Model I: \( \rho_g = 1, \rho_z = 1 \)
Model III: $\rho_g = 1$, $\rho_z$ estimated
Model III: $\rho_g = 1, \rho_z$ estimated

Posterior Distribution of $\rho_z$ in Model III
Model I vs. Model III

- The difference between the median path of \( r^* \) in Model I and Model II doesn’t seem large.
- Model III looks different than Model I (in terms of the median path).
- The only difference between the models is that Model I has a degenerate prior on \( \rho_z \equiv 1 \).

Model Comparison

- Bayes Factor for nested models reduces to the Savage-Dickey density ratio (Dickey, 1971).

\[
B_{III,I} = \frac{pr(Y|M_{III})}{pr(Y|M_I)} = \frac{p_{III}(\rho_z = 1)}{p_{III}(\rho_z = 1|Y)}
\]
Savage-Dickey Density Ratio

Figure: Prior and Post. $\rho_z$

$$B_{III,I} = \frac{p_{III}(\rho_z = 1)}{p_{III}(\rho_z = 1 | Y)} = 8.4$$
## Model Comparison

<table>
<thead>
<tr>
<th></th>
<th>Bayesian</th>
<th></th>
<th>MLE</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Log Marg. Like.</td>
<td>BF</td>
<td>Log. Like.</td>
<td>BIC</td>
</tr>
<tr>
<td>Baseline</td>
<td>-533</td>
<td>0.1</td>
<td>-518</td>
<td>1088</td>
</tr>
<tr>
<td>Alternative</td>
<td>-526</td>
<td>10.2</td>
<td>-517</td>
<td>1093</td>
</tr>
</tbody>
</table>
Connection to Theory

- The $r^*$ equation is a linearized euler equation, we can denote the stochastic discount factor (SDF) by $S$:

$$e^{-r_t^*} = E_t[S_{t+1}]$$

- Consider an SDF that differs from that of log-utility by an extra term ($\tilde{Z}$) as in Campbell and Cochrane, Epstein-Zin, etc. Then we have

$$r_t^* = \log E_t \left[ \frac{C_{t+1}}{C_t} \tilde{Z}_{t+1} \right] = \log E_t \left[ e^{g_{t+1} + \tilde{z}_{t+1}} \right] \approx E_t [g_{t+1} + z_{t+1}]$$

- $z_t$ can be interpreted as an asset pricing term that measures the separation from log-utility of the SDF. We can give $z$ this “headwinds” interpretation.
Headwinds

- Frequently, “headwinds” are cited as a reason for why the level of $r^*$ is still so low.
  - “…lingering sense of caution on the part of households and businesses in the wake of the trauma of the Great Recession.” (Yellen, 3/3/17)
- $z_t$ is the “special sauce” (Williams, 2015 Brookings), it is all the things that are not economic growth.
  - There is nothing that says the components have to be stationary or persistent.
  - In the current version of the model, there is no data specifically aimed at estimating $z_t$.
  - $z_t$ soaks up the variation in the rate gap that doesn’t appear to be linked to growth.
  - Headwinds seem like they should be temporary.
Single-step Bayesian estimation with less informative priors shows deeper drops and subsequent recoveries after recessions, in contrast to multi-step MLE results.

When $z$ is not assumed to be a random walk, we estimate a greater recovery of $r^*$ since the lows of the great recession, reaching closer to 2% at the end of 2016Q3.

Our conclusion is that permanent shocks to $z$ (and thus, in our minds, the SDF) are needed to produce a persistent low level of $r^*$ after the great recession.

The dynamics of $z$ are hard to estimate with this data.

  More structure around $z$ may be helpful.
APPENDIX
## Parameter Estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Bayesian Baseline</th>
<th>Bayesian Alternative</th>
<th>MLE Baseline</th>
<th>MLE Alternative</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>1.251 [0.97,1.51]</td>
<td>1.270 [0.84,1.52]</td>
<td>1.531 [1.36,1.70]</td>
<td>1.530 [1.36,1.70]</td>
</tr>
<tr>
<td>(a_2)</td>
<td>-0.364 [-0.58,-0.07]</td>
<td>-0.348 [-0.59,0.05]</td>
<td>-0.589 [-0.76,-0.42]</td>
<td>-0.587 [-0.76,-0.41]</td>
</tr>
<tr>
<td>(a_r)</td>
<td>-0.113 [-0.19,-0.06]</td>
<td>-0.093 [-0.18,-0.06]</td>
<td>-0.070 [-0.10,-0.04]</td>
<td>-0.067 [-0.10,-0.04]</td>
</tr>
<tr>
<td>(b_1)</td>
<td>0.682 [0.57,0.79]</td>
<td>0.665 [0.56,0.78]</td>
<td>0.671 [0.60,0.74]</td>
<td>0.670 [0.60,0.74]</td>
</tr>
<tr>
<td>(b_Y)</td>
<td>0.051 [0.03,0.13]</td>
<td>0.071 [0.04,0.15]</td>
<td>0.077 [0.04,0.12]</td>
<td>0.079 [0.04,0.12]</td>
</tr>
<tr>
<td>(\sigma_1)</td>
<td>0.412 [0.11,0.66]</td>
<td>0.279 [0.08,0.57]</td>
<td>0.355 [0.21,0.50]</td>
<td>0.365 [0.21,0.52]</td>
</tr>
<tr>
<td>(\sigma_2)</td>
<td>0.794 [0.74,0.87]</td>
<td>0.795 [0.74,0.86]</td>
<td>0.791 [0.75,0.83]</td>
<td>0.791 [0.75,0.83]</td>
</tr>
<tr>
<td>(\sigma_3)</td>
<td>0.149 [0.07,1.69]</td>
<td>1.755 [0.67,3.95]</td>
<td>0.160 [0.10,0.23]</td>
<td>0.172 [0.10,0.25]</td>
</tr>
<tr>
<td>(\sigma_4)</td>
<td>0.564 [0.1,0.64]</td>
<td>0.580 [0.25,0.65]</td>
<td>0.571 [0.48,0.66]</td>
<td>0.567 [0.47,0.66]</td>
</tr>
<tr>
<td>(\sigma_5)</td>
<td>0.036 [0.02,0.13]</td>
<td>0.035 [0.02,0.11]</td>
<td>0.030 [0.02,0.03]</td>
<td>0.030 [0.02,0.03]</td>
</tr>
<tr>
<td>(\rho_z)</td>
<td>1*</td>
<td>0.789 [0.31,0.89]</td>
<td>1*</td>
<td>0.916 [0.77,1.06]</td>
</tr>
</tbody>
</table>
Laubach-Williams Methodology

Laubach-Williams 3-part Estimation

Step 1  Hold $g$ constant, drop real rate gap from model, then:
- Get estimate of potential output, $\hat{y}^*$, compute $\Delta \hat{y}^*$
- $\lambda_g$ is equal to Andrews and Ploberger (1994) exponential Wald statistic for the test of a structural break at unknown date in $\Delta \hat{y}^*$.

Step 2  Impose $\lambda_g$ value from Step 1, include real rate gap, but hold $z$ constant, then:
- Estimate the simplified model
- $\lambda_z$ is equal to Wald statistic for the test of a shift in the intercept of the IS equation.

Step 3  Impose $\lambda_g$ from Step 1 and $\lambda_z$ from Step 2, and estimate the remaining parameters by MLE.
HLW replication

One-sided Bayesian estimate (blue) and HLW results (red)
DeJong and Whiteman (1993)

Monte Carlo exercise where the true parameter value is 0.85, $T = 100$
Sample distribution of MLE estimate (left) and posterior distribution (right)
Our estimation technique does not suffer from the pile-up problem.

To illustrate this: Consider Stock and Watson 1998 local level model of log GDP growth.

\[
\Delta y_t = \beta_t + u_t \\
\beta_t = \beta_{t-1} + \frac{\lambda}{T} \eta_t \\
u_t = a_1 u_{t-1} + a_2 u_{t-2} + a_3 u_{t-3} + a_4 u_{t-4} + \varepsilon_t
\]
Getting Around the “pile-up” problem

Replication Stock and Watson 98

Histogram of draws of lambda

Lewis & Vazquez-Grande (FRB)

Measuring $r^*$
Getting Around the “pile-up” problem

Replication Stock and Watson 98

![Chart showing dlGDP and beta over time from 1950 to 2030]
Key Posterior Estimates From Each Model
Model I: $\rho_g = 1, \rho_z = 1$
Model I: $\rho_g = 1, \rho_z = 1$

Smoothed $g$ path
Model I: $\rho_g = 1, \rho_z = 1$
Model I: $\rho_g = 1, \rho_z = 1$

Posterior of $\sigma_{r^*}$
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$

Smoothed output gap
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$

Posterior of $\rho_g$
Model II: $\rho_g$ and $\mu_g$ estimated, $\rho_z = 1$

Posterior of $\mu_g$ (Quarterly)
Model III: $\rho_g = 1, \rho_z$ estimated

Smoothed output gap
Model III: \( \rho_g = 1, \rho_z \) estimated

Smoothed \( g \) path
Model III: $\rho_g = 1$, $\rho_z$ estimated
Model III: $\rho_g = 1$, $\rho_z$ estimated
Model III: $\rho_g = 1, \rho_z$ estimated

Posterior of $\rho_z$
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated

Smoothed output gap
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated

Smoothed $g$ path
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated

Smoothed $z$ path
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated

Posterior of $\sigma_{r^*}$
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated

Posterior of $\rho_g$
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated

Posterior of $\mu_g$ (Quarterly)
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated

Posterior of $a_r$
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated
Model IV: $\rho_g$, $\mu_g$ and $\rho_z$ estimated

Posterior of $b_1$