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Restricted Complementarity and Paths to Stability in Matching with Couples*

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Abstract

We study matching with couples problems where hospitals have one vacant position. We introduce a constraint on couples' preferences over pairs of hospitals called restricted complementarity, which is a "translation" of bilateral substitutability in matching with contracts. Next, we extend Klaus and Klijn's (2007) path to stability result by showing that if couples' preferences satisfy restricted complementarity, then from any arbitrary matching, there exists a finite path of matchings where each matching on the path is obtained by "satisfying" a blocking coalition for the previous one and the final matching is stable.

Keywords: Matching, Couples, Paths, Stability, Restricted Complementarity.
JEL classification: C78, D47.

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1 Introduction

A matching with couples problem is a mathematical representation of a labor market with two salient features: (i) wages are fixed, and hence they cannot be used to equate labor supply and demand, and (ii) married couples participate in the market. An example of such a labor market is the entry-level labor market for medical doctors in the e U.S. which is administered by the National Resident Matching Program (NRMP).

Since the 1950s, the NRMP has used a variant of the Gale and Shapley (1962) algorithm to match doctors and hospitals. It was by 1970 that the increasing presence of married couples in the market led to a significant reduction of voluntary participation in the NRMP. This problem was tackled by allowing couples to express their preferences over pairs of hospitals.[1] The difficulties the NRMP experienced before its redesign suggest market outcomes were not "stable" in the way we describe next.

A blocking coalition consists of a group of doctors and hospitals that are not matched to each other but would prefer to be. A matching is stable if there is no blocking coalition. In the presence of blocking coalitions, the permanence of the matching is at serious risk as there are agents who have the incentive and the power to circumvent it.

Gale and Shapley (1962) demonstrate that there is always a stable matching in matching problems with no couples. Unfortunately, in the presence of couples, a stable matching is no longer guaranteed (Roth, 1984).

The success of matching markets with couples such as the NRMP suggests that, despite the theoretical impossibility, stable matchings exist and are reached in real-life applications. These findings have led to investigating properties of couples' preferences that guarantee the existence of stable matchings. Some such properties are weak responsiveness (Klaus and Klijn, 2005 and Klaus et al., 2009), substitutability (Hatfield and Milgrom, 2005), and bilateral substitutability (Hatfield and Kojima, 2010). Bilateral substitutability is weaker than both substitutability and weak responsiveness (Hatfield and Kojima, 2010).[2]

In this paper, we introduce a property of couples' preferences over pairs of hospitals called restricted complementarity. Restricted complementarity is the "translation" of bilateral substitutability from the matching with contracts model to the standard couples

[1] See Roth and Peranson (1999) and Roth (2002) for details on the new design of the algorithm.

[2] For an interdisciplinary and comprehensive review of the literature on matching with couples problems, see Biró and Klijn (2013).

model. Tello (2016) shows that, in the standard couples model, bilateral substitutability is the minimal restriction on couples' preferences for the existence of stable matchings, and therefore is restricted complementarity. Furthermore, restricted complementarity being a condition on preference orderings over pairs of hospitals, instead of a condition over choice functions, makes it possible to adapt techniques introduced by Roth and Vande Vate (1990) and Klaus and Klijn (2007).

Our main result is that for problems where all couples' preferences satisfy restricted complementarity, it is possible to reach a stable matching from any arbitrary matching by satisfying blocking coalitions one by one.[3] This result implies that starting from an arbitrary matching, certain random processes that match blocking coalitions converge to a stable matching with probability one.[4]

The importance of this result is that it provides theoretical support to the empirical observation that many decentralized matching markets perform well, suggesting they can reach stable outcomes.

The remainder of the paper is organized as follows. Section 2 describes the matching with couples problem and introduces restricted complementarity. In Section 3, we give the path to stability result. We conclude in Section 4. All proofs are in the Appendices.

2 Matching with couples

There are two finite sets $\boldsymbol{H}$ and $\boldsymbol{C}$ of **hospitals** and **couples**. We denote generic elements of H and C by h and $c = (d_1, d_2)$, where d_1 and d_2 denote the spouses in a couple c. Let $\boldsymbol{D} := \{d : d \in \{d_1, d_2\} \text{ for some } (d_1, d_2) \in C\}$ be the set of **doctors**. Each hospital has exactly one position to fill. Let $\boldsymbol{u}$ be the **outside option** for doctors. We can think of u as a hospital with no capacity constraint so that each doctor can always find a job there.[5]

Each hospital $h \in H$ has a complete, transitive, and strict **preference relation** $\boldsymbol{P}_h$ over the set D, and the prospect of having its position unfilled denoted by $\emptyset$. For $d, d' \in$

[3] Satisfying a blocking coalition: a couple or a hospital ends its partnership with unacceptable partners, or a couple and two hospitals match with each other, possibly in replacement of less preferred partners.

[4] This result is established for one-to-one matching problems by Roth and Vande Vate (1990). It is extended to many-to-many matching problems in which agents on one side of the market have substitutable preferences and agents on the other side have responsive preferences by Kojima and Ünver (2006). For matching with couples problems, the path to stability result holds whenever couples' preferences satisfy weak responsiveness (Klaus and Klijn, 2007).

[5] We refer to elements of $H \cup \{u\}$ as hospitals. When we refer only to elements in H we make it explicit by writing "hospitals in H".

$D \cup \{\emptyset\}$, we write $d\, P_h\, d'$ if hospital h prefers d to d' ($d \neq d'$), and $d\, \boldsymbol{R_h}\, d'$ if h finds d at least as good as d', i.e., $d\, P_h\, d'$ or $d = d'$. If $d \in D$ is such that $d\, P_h\, \emptyset$, then d is an **acceptable doctor** for hospital h. By contrast, if $\emptyset\, P_h\, d$, d is an unacceptable doctor for hospital h.

We represent hospitals' preferences by ordered lists of doctors and $\emptyset$; for example, $P_h = d_5, d_3, \emptyset \ldots$ indicates that hospital h prefers d_5 to d_3, and considers all other doctors to be unacceptable. Let $\boldsymbol{P_H} = \{P_h\}_{h \in H}$.

The restriction that each hospital has exactly one vacant position implies that no couple can get a job for each of its members in the same hospital. In other words, no pair (h, h) with $h \in H$ is **feasible**. The set of all feasible hospital pairs is given by

$$\boldsymbol{\bar{\mathcal{H}}} = [(H \cup \{u\}) \times (H \cup \{u\})] \setminus \{(h, h) : h \in H\}.$$

We denote a generic element of $\bar{\mathcal{H}}$ by (h, h').

Each couple $c = (d_1, d_2) \in C$ has a complete, transitive, and strict **preference relation** $\boldsymbol{P_c}$ over $\bar{\mathcal{H}}$. For each $(h_1, h_2), (h_3, h_4) \in \bar{\mathcal{H}}$ we write $(h_1, h_2)\, P_c\, (h_3, h_4)$ if c prefers d_1 and d_2 being matched to h_1 and h_2 respectively, to being matched to h_3 and h_4 respectively. We write $(h_1, h_2)\, \boldsymbol{R}_c\, (h_3, h_4)$ if c finds (h_1, h_2) at least as good as (h_3, h_4), i.e., $(h_1, h_2)\, P_c\, (h_3, h_4)$ or $(h_1, h_2) = (h_3, h_4)$. Any pair (h, h') such that $(h, h')\, R_c\, (u, u)$ is an **acceptable pair** to c and otherwise unacceptable.

We represent couples' preferences by means of ordered lists of feasible hospital pairs; for example, $P_c = (h_3, h_4), (h_5, h_3), (u, h_4), \ldots, (u, u) \ldots$ indicates that c prefers (h_3, h_4) to (h_5, h_3) and so on. Let $\boldsymbol{P_C} = \{P_c\}_{c \in C}$.

A one-to-one matching with couples problem or simply a **problem** is denoted by $\boldsymbol{(P_H, P_C)}$.

For each c we define a **choice function** Ch_c as

$$\boldsymbol{\mathsf{Ch}_c(\mathcal{H})} := \overset{\text{argmax}}{\scriptstyle P_c} \big\{\mathcal{H} \cup \{(u, u)\}\big\}, \quad \text{for each} \;\; \mathcal{H} \subseteq \bar{\mathcal{H}}.$$

The choice function is defined on the set of feasible hospital pairs. Given a feasible set of hospital pairs, it selects the most preferred pair from the set and the outside option (u, u). It is important to highlight that choice functions are not a primitives of our model.

A **matching** specifies which hospitals are matched to which doctors. Formally, a matching $\boldsymbol{\mu}$ is a function defined on $D \cup H$ such that

- for each $d \in D$, $\mu(d) \in H \cup \{u\}$,

- for each $h \in H$, $\mu(h) \in D \cup \{\emptyset\}$,
- for each $d \in D$ and $h \in H$, $\mu(d) = h$ if and only if $\mu(h) = d$.

For each $c = (d_1, d_2) \in C$, we write $\boldsymbol{\mu(c)}$ to denote the pair $(\mu(d_1), \mu(d_2))$.

Now we introduce a central property of the matching literature: stability. Our stability concept is the same as the one in Klaus and Klijn (2005).

Let μ be a matching. A coalition $[\boldsymbol{h}]$ with $h \in H$ is a **blocking hospital** for μ if

- $\emptyset \, P_h \, \mu(h)$.

Let $c = (d_1, d_2) \in C$. A coalition $[\boldsymbol{c, (u, u)}]$, $[\boldsymbol{c, (\mu(d_1), u)}]$ or $[\boldsymbol{c, (u, \mu(d_2))}]$ is a **blocking couple** for μ if

- $\big(u, u\big) \, P_c \, \big(\mu(d_1), \mu(d_2)\big)$, $\big(\mu(d_1), u\big) \, P_c \, \big(\mu(d_1), \mu(d_2)\big)$ or $\big(u, \mu(d_2)\big) \, P_c \, \big(\mu(d_1), \mu(d_2)\big)$, respectively.

We often refer to blocking hospitals and blocking couples as blocking coalitions.

A coalition $[\boldsymbol{c, (h, h')}]$ with $(h, h') \in \bar{\mathcal{H}}$ is a **blocking coalition** for μ if $(h, h') \notin \{(u, u), \big(\mu(d_1), u\big), \big(u, \mu(d_2)\big)\}$ and

- $\big(h, h'\big) \, P_c \, \big(\mu(d_1), \mu(d_2)\big)$;
- $[h \in H$ implies $d_1 \, R_h \, \mu(h)]$ and $[h' \in H$ implies $d_2 \, R_{h'} \, \mu(h')]$.

A matching is **stable** if there are no blocking coalitions. Since our analysis focuses on stability, whenever we specify a problem (P_H, P_C) it is enough to specify lists of acceptable doctors and lists of acceptable (and feasible) hospital pairs.

A set of hospital pairs is complete if (i) it contains the pair (u, u), and (ii) if combining the first and second components of any two pairs within the set results in a feasible hospital pair, then the latter pair also belongs to the set. Formally:

A subset $\mathcal{H} \subseteq \bar{\mathcal{H}}$ is **complete** if (i) $(u, u) \in \mathcal{H}$ and (ii) $\big[(h_1, h_2), (h_3, h_4) \in \mathcal{H}$ and $(h_1, h_4) \in \bar{\mathcal{H}}\big]$ imply $(h_1, h_4) \in \mathcal{H}$.

We define restricted complementarity, which is a property of couples' preferences P_c over hospital pairs.

Restricted complementarity: for each complete $\mathcal{H} \subseteq \bar{\mathcal{H}}$ and each h_1, h_2, h_3, h_4 such that $h_1, h_2 \notin \{u, h_3, h_4\}$, $(h_3, h_4) \in \mathcal{H}$, and $\mathsf{Ch}_c(\mathcal{H}) = (h_1, h_2)$, we have

$$\big[\textbf{(sc1)}\ (h_1, h_4)\, P_c\, (h_3, h_4) \quad \text{or} \quad \textbf{(sc2)}\ (h_1, u)\, P_c\, (h_3, h_4)\big]$$

and

$$\big[\textbf{(sc3)}\ (h_3, h_2)\, P_c\, (h_3, h_4) \quad \text{or} \quad \textbf{(sc4)}\ (u, h_2)\, P_c\, (h_3, h_4)\big].$$

We explain *restricted complementarity* by means of an example. Suppose married doctors A and B receive several offers from hospitals h_1, h_2, h_3, h_4. Suppose that they take the offer from hospital h_1 for A and the offer from the hospital h_2 for B. Let $h_3 \neq h_1, h_2$ be a hospital that made an offer to A. Further, suppose that the offer from hospital h_2 is no longer available to B. In this case A rejecting the offer from h_1 and taking the offer from h_3 while B taking an offer from a hospital $h_4 \neq h_1$ is a violation of *restricted complementarity.* However, A taking the offer from h_3 and B taking the offer from $h_4 = h_1$ is not a violation of *restricted complementarity.* Complementarities that involve this kind of job swaps between couple's spouses are allowed under restricted complementarity.

Proposition 1. Let (P_H, P_C) be a problem such that for each $c \in C$, P_c satisfies restricted complementary, then a stable matching exist.

The proof to Proposition 1 is relegated to Appendix A.

In Appendix A, we define bilateral substitutability. Tello 2016 shows that in the couples model, bilateral substitutability is the minimal condition for which the existence of stable matchings is guaranteed. We prove Proposition 1 by showing that restricted complementarity is equivalent to bilateral substitutability. The advantage of working with restricted complementarity is that it is possible to extend the techniques introduced by Roth and Vande Vate (1990) and Klaus and Klijn (2007).

3 Results

We extend Klaus and Klijn's (2007) path to stability result to the case when couples' preferences satisfy *restricted complementarity.* We show that if all couples' preferences satisfy *restricted complementarity*, there is always a path from an arbitrary matching to a stable one, such that each matching on the path is obtained by "satisfying" a blocking coalition for the previous matching. We first define precisely what we mean by "satisfying" a blocking coalition.

Satisfying blocking coalitions:[6] If $[h]$, $h \in H$ is a blocking hospital for a matching μ, then we say that a new matching ν is obtained from μ by satisfying the blocking coalition if h and $\mu(h)$ are unmatched, and all other agents are matched to the same mates at ν as they are at μ. Formally, matching ν is obtained from matching μ by satisfying blocking coalition $[h]$ for μ if

- $\nu(h) = \emptyset$ and $\nu(\mu(h)) = u$;
- $\nu(d) = \mu(d)$ for each $d \in D \setminus \{\mu(h)\}$;
- $\nu(\bar{h}) = \mu(\bar{h})$ for each $\bar{h} \in H \setminus \{h\}$.

Similarly, if $[c = (d_1, d_2), (h', h'')]$ is a blocking couple or a blocking coalition for a matching μ, then we say that a new matching ν is obtained from μ by satisfying the blocking coalition if (d_1, d_2) and (h, h') are matched to one another at ν, their mates at μ (if any, and if not involved in the blocking coalition) are unmatched at ν, and all other agents are matched to the same mates at ν as they were at μ. Formally, matching ν is obtained from matching μ by satisfying blocking coalition $[(d_1, d_2), (h', h'')]$ (for μ) if

- $\big[\mu(d_1) = h \in H \setminus \{h', h''\}$ implies $\nu(h) = \emptyset\big]$ and $\big[\mu(d_2) = h \in H \setminus \{h', h''\}$ implies $\nu(h) = \emptyset\big]$;
- $\big[\mu(h') = d \in D \setminus \{d_1, d_2\}$ implies $\nu(d) = u\big]$ and $\big[\mu(h'') = d \in D \setminus \{d_1, d_2\}$ implies $\nu(d) = u\big]$;
- $\nu(d_1) = h'$, $\nu(d_2) = h''$, $\big[h' \in H$ implies $\nu(h') = d_1\big]$, and $\big[h'' \in H$ implies $\nu(h'') = d_2\big]$;
- $\nu(d) = \mu(d)$ for each $d \in D \setminus \{\mu(h'), \mu(h''), d_1, d_2\}$;
- $\nu(h) = \mu(h)$ for each $h \in H \setminus \{\mu(d_1), \mu(d_2), h', h''\}$. $\Diamond$

Now we are ready to state our path to stability result.

Theorem 1 (*Paths to stability result*)**.** *Let (P_H, P_C) be a problem such that for each $c \in C$, P_c satisfies restricted complementarity. Let μ be an arbitrary matching for (P_H, P_C).*

[6] We borrow the definition of satisfying blocking coalitions from Klaus and Klijn (2007, page 159).

Then, there is a finite sequence of matchings $\mu_1, \ldots, \mu_k$ such that $\mu_1 = \mu$, μ_k is stable, and for each $i = 1, \ldots, k-1$, there is a blocking coalition for μ_i such that μ_{i+1} is obtained from μ_i by satisfying this blocking coalition.

The proof to Theorem 1 is relegated to Appendix B.

As a Corollary to Theorem 1 we obtain the following result. Consider a random process that begins by selecting an arbitrary matching μ and generates the sequence of matchings $\mu = \mu_1, \mu_2, \ldots$ where each μ_{i+1} is obtained from μ_i by satisfying a blocking coalition, chosen at random from the blocking coalitions for μ_i. Assume that the probability that any particular blocking coalition for μ_i is chosen to generate μ_{i+1} is positive, and only depends on the matching μ_i (but not on the number i). Let $\Psi(\mu)$ be the random sequence generated in this way from an initial matching μ.

Corollary 1 (*Random paths to stability*)**.** *Let (P_H, P_C) be a problem such that for each $c \in C$, P_c satisfies restricted complementarity. For any initial matching μ for (P_H, P_C), the random sequence $\Psi(\mu)$ converges with probability one to a stable matching.*

To prove Theorem 2, we adapt the deterministic path algorithm to stability from Klaus and Klijn's (2007) DPC-Algorithm.[7] Our algorithm yields, in a finite number of steps, a stable matching for any problem in which couples' preferences satisfy *restricted complementarity.*

In the description of our algorithm we use the aid of a virtual room that agents enter and exit throughout the algorithm. This didactic visualization was first introduced by Ma (1996) and is also used in Klaus and Klijn (2007).

3.1 Paths to Stability Algorithm (PS-algorithm)

Let μ be an arbitrary matching for a problem (P_H, P_C) where for each $c \in C$, P_c satisfies restricted complementarity.[8] After satisfying blocking hospitals for μ (first stage) we start putting couples one by one in an initially empty room (second stage). Each couple enters the room with its mates under μ. Whenever a couple enters the room with its mates, blocking coalitions within the room are satisfied and the hospitals that are "dumped" are put outside the room. Thus, after this second stage we obtain a matching where all couples

[7] The DPC-Algorithm of Klaus and Klijn (2007) is in turn a modification of the Roth and Vande Vate (1990) algorithm for one-to-one matching problems.

[8] This subsection follows closely Klaus and Klijn (2007, pages 161-163).

are matched to hospitals in the room, and for which there are no blocking coalitions within the room.[9] In the third stage, we let hospitals outside the room enter one by one. In each step possibly one blocking coalition within the room has to be satisfied before turning to the next step. The blocking coalitions that are satisfied in this stage are "hospital optimal" in the sense that for the hospital involved there is no other blocking coalition available within the room that would give it a better doctor. We call the doctor that is in all hospital optimal blocking coalitions associated with the entering hospital the *best doctor*. There may be several blocking coalitions that match the entering hospital with the best doctor. In order to assure the convergence of the algorithm we have to choose the blocking carefully. First, we prove (see the Claim in the third stage of the PS-algorithm and its proof in Appendix B) that one of the following is a blocking coalition: (a) the couple (to which the best doctor belongs), the hospital and the match of the best doctor's partner, (b) the couple, the hospital and the best doctor's match, or (c) the couple and the hospital. From these possible blocking coalitions we satisfy the blocking coalition that the couple prefers most. In the process of satisfying the blocking coalition at most two hospitals may exit the room.

We show that after a finite number of steps all hospitals have joined the couples in the room. Starting from μ we have obtained a stable matching for the problem (P_H, P_C). We now formalize the PS-algorithm.

A formal description of the PS-algorithm

Input: A problem (P_H, P_C) such that for each $c \in C$, P_c satisfies restricted complementarity, and a matching μ for (P_H, P_C).
Initialization: Set $A := \emptyset$. We call A the room.

- **First Stage**
  - Satisfy all blocking hospitals and blocking couples and denote the resulting matching by μ. `After Stage 1 we obtain a matching` $\mu^1 := \mu$ `with no blocking hospitals/couples.`
- **Second Stage**

[9]Up to stages 1 and 2 the PS-algorithm is exactly the same as the DPC-Algorithm of Klaus and Klijn (2007, pages 161-163). It is in the third stage where an adaptation is needed to deal with preferences that satisfy *restricted complementarity* but do not satisfy weak responsiveness.

– If there is $c = (d_1, d_2) \in C \setminus A$, then let the couple and the hospitals in H assigned to it enter the room, i.e., set $A := (A \cup \{c, \mu(d_1), \mu(d_2)\}) \setminus \{u\}$.

– As long as there is a blocking coalition $[c' = (d'_1, d'_2), (h'_1, h'_2)]$ with $\{c', h'_1, h'_2\} \subseteq A \cup \{u\}$ do:

Begin Loop: Satisfy $[c', (h'_1, h'_2)]$, and let dumped hospitals exit the room:

* For $i = 1, 2$, [if $\mu(d'_i) = h \in H \setminus \{h'_1, h'_2\}$], then define $\mu(h) := \emptyset$ and set $A := A \setminus \{h\}$;
* For $i = 1, 2$, if $h'_i \in H$ and $\mu(h'_i) = d \in D \setminus \{d'_1, d'_2\}$, then $\mu(d) := u$;
* For $i = 1, 2$, define $\mu(d'_i) := h'_i$, and if $h'_i \in H$, then $\mu(h'_i) := d'_i$.

End Loop

`After Stage 2 we obtain a matching` $\mu^2 := \mu$ `where all couples are in the room and there is no blocking coalitions.`

- **Third Stage**

– As long as there is $h' \in H \setminus A$ do:

Begin Loop: Set $A := A \cup \{h'\}$.

If there is no blocking coalition $[c', (h'_1, h'_2)]$ with $h' \in \{h'_1, h'_2\} \subseteq A \cup \{u\}$, then GO BACK to the beginning of the Third Stage. If there are blocking coalitions $[c', (h'_1, h'_2)]$ with $h' \in \{h'_1, h'_2\} \subseteq A \cup \{u\}$, then let d'_1 be h''s most preferred doctor among the ones it could be matched to at these blocking coalitions. Let d'_2 be the partner of d'_1. Without loss of generality, $c' = (d'_1, d'_2) \in C$.

Let $\boldsymbol{h_1^*} = \mu(d'_1)$, $\boldsymbol{h_2^*} = \mu(d'_2)$.

Claim: $[c', (h', h_2^*)]$, $[c', (h', h_1^*)]$ or $[c', (h', u)]$ is a blocking coalition for μ.

For each couple $c \in C$ and each matching ν, let

$$\mathcal{B}(c, \nu) := \big\{(h_i, h_j) \in \bar{\mathcal{H}} : \ [c, (h_i, h_j)] \text{ is a blocking coalition for } \nu\big\}.^{10}$$

Define

$$(h', \hat{h}) = \mathsf{Ch}_{c'}\Big\{\{(h', h_2^*), (h', h_1^*), (h', u)\} \cap \mathcal{B}(c', \mu)\Big\}.$$

The intersection above is non-empty by the Claim. Satisfy blocking coalition $[c', (h', \hat{h})]$, and if some hospitals are dumped (at most two), let them exit the

[10]This is the set of all hospital pairs that together with c form a blocking coalition for matching ν.

room. Formally,
define $\mu(c') := (h', \hat{h})$ and,

- $*$ **Case (a)** If $\hat{h} = h_2^*$ and $h_1^* \in H$, then define $\mu(h_1^*) = \emptyset$ and set $A := A \setminus \{h_1^*\}$.
- $*$ **Case (b)** If $\hat{h} = h_1^*$ and $h_2^* \in H$, then define $\mu(h_2^*) = \emptyset$ and set $A := A \setminus \{h_2^*\}$.
- $*$ **Case (c)** If $\hat{h} = u$, then for each $h^* \in \{h_1^*, h_2^*\} \cap H$, define $\mu(h^*) = \emptyset$ and set $A := A \setminus \{h_1^*, h_2^*\}$.

End Loop

`After Stage 3 we obtain a matching` $\mu^3 := \mu$ `where all couples and all hospitals are in the room and no blocking coalitions exist in the room.`

Output: A stable matching μ for (P_H, P_C).

Remark 1. One may wonder whether for any problem for which a stable matching exists, there exists some algorithm that starts in an arbitrary matching and converges to a stable one. The answer to this question is negative. This means that there are problems for which the set of stable matching is non-empty and no path of matchings obtained by satisfying blocking coalitions and starting from certain matching converges to a stable one. Example 4.1 of Klaus and Klijn (2007, page 167) exhibits a problem for which a stable matching exists and, starting from a certain matching, any path obtained by satisfying blocking coalitions cycles. As Klaus and Klijn (2007) point out: "this cycling has to do with the underlying complementarities in the couples' preferences, and not with the particular choice of the path (algorithm)."

Remark 2. The path to stability result generalizes to problems with couples and single doctors. We can incorporate single doctors by letting each single doctor have a fictitious partner that finds all hospitals unacceptable. For example, if single student d has preferences given by $P_d = h_1, h_2, u, h_4, \ldots$ then replace d by couple c with preferences $P_c = (h_1, u), (h_2, u), (u, u), (h_4, u) \ldots$ The path convergence result generalizes because the preferences of a fictitious couple induced by the preferences of a single doctor satisfy *restricted complementarity*.

Example 1 (An application of the PS-algorithm)**.** Let (P_H, P_C) be the problem in Table 1. The sets of hospitals and couples are given by $H = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8\}$ and $C = \{(d_1, d_2), (d_3, d_4), (d_5, d_6), (d_7, d_8), (d_9, d_{10})\}$. All couples' preferences satisfy *restricted* *complementarity*. In Table 2 we give a path of matchings $\mu_0, \ldots, \mu_{10}$ such that $\mu_0(C) = (u, h_4), (h_5, u), (h_8, h_3), (h_2, h_1), (h_6, h_7)$ is unstable, each matching on the path is obtained from the previous matching by satisfying a blocking coalition, and the final matching μ_{10} is stable.

We obtain such a path by applying the PS-algorithm to (P_H, P_C) and the initial matching μ_0. In Table 3 we can follow the execution of the PS-algorithm. Table 3 shows several items at each step of the algorithm: the matching, the agents that enter the room, the agents that exit the room, and the blocking coalitions that are satisfied. We abbreviate the term blocking coalition by b.c.

For the PS-algorithm to work it has to be that at some point all couples and hospitals are in the room. In Table 3 it is easy to see that at step 8 the number of agents in the room decreases, as only one agent enters and two agents exit. We may be concerned that the algorithm cycles with agents entering and exiting the room. This is not the case as we show in Appendix B.

Table 1: A problem where couples' preferences satisfy *restricted complementarity*

P_C					P_H							
d_1d_2	d_3d_4	d_5d_6	d_7d_8	d_9d_{10}	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
h_3u	uh_1	h_6u	h_5h_2	h_6h_7	d_1	d_3	d_1	d_2	d_6	d_5	d_1	d_5
uh_2	h_2h_1	uh_5	h_5u	h_7h_6	d_4	d_1	d_2	d_1	d_8	d_9	d_5	
h_1h_4	h_2u	h_8u	uh_2	h_6u	d_3	d_2	d_3	d_3	d_7		d_9	
h_1u	h_5u		h_2u	h_7u	d_2	d_7	d_4	d_8	d_1			
uh_4				uh_7	d_5	d_8	d_6	d_5	d_2			
				uh_6				d_6	d_3			

Table 2: A path to stability for problem (P_H, P_C)

μ's/couples	d_1d_2	d_3d_4	d_5d_6	d_7d_8	d_9d_{10}	$\emptyset\ \emptyset\ \emptyset\ \emptyset\ \emptyset$
μ_0	uh_4	h_5u	h_8h_3	h_2h_1	h_6h_7	
μ_1	uh_4	h_5u	h_8u	h_2u	h_6u	h_1, h_3, h_7
μ_2	uh_4	uu	uh_5	h_2u	h_6u	h_1, h_3, h_7, h_8
μ_3	uh_4	h_2u	uh_5	uu	h_6u	h_1, h_3, h_7, h_8
μ_4	uh_4	h_2u	h_6u	uu	uu	h_1, h_3, h_5, h_7, h_8
μ_5	h_1h_4	h_2u	h_6u	uu	uu	h_3, h_5, h_7, h_8
μ_6	h_3u	h_2u	h_6u	uu	uu	h_1, h_4, h_5, h_7, h_8
μ_7	h_3u	uh_1	h_6u	uu	uu	h_2, h_4, h_5, h_7, h_8
μ_8	h_3u	uh_1	h_6u	h_2u	uu	h_4, h_5, h_7, h_8
μ_9	h_3u	uh_1	h_6u	h_5h_2	uu	h_4, h_7, h_8
μ_{10}	h_3u	uh_1	h_6u	h_5h_2	h_7u	h_4, h_8

Table 3: PS-algorithm step by step

Step	Stage	Matching	Room	Satisfy b.c.s in this column	Output
1	1	μ_0	$\emptyset$	blocking couples and blocking hospitals	μ_1
			Enter room	b.c. within the room	Exit room
2	2	μ_1	$(d_1, d_2),\ h_4$	-	-
3	2	μ_1	$(d_3, d_4),\ h_5$	-	-
4	2	μ_1	$(d_5, d_6),\ h_8$	$[(d_5d_6),(uh_5)]$	h_8
5	2	μ_2	$(d_7, d_8),\ h_2$	$[(d_3d_4),(h_2u)]$	-
6	2	μ_3	$(d_9, d_{10}),\ h_6$	$[(d_5d_6),(h_6u)]$	h_5
7	2	μ_4	h_1	$[(d_1d_2),(h_1h_4)]$	-
8	3	μ_5	h_3	$[(d_1d_2),(h_3u)]$	h_1, h_4
9	3	μ_6	h_1	$[(d_3d_4),(uh_1)]$	h_2
10	3	μ_7	h_2	$[(d_7d_8),(h_2u)]$	-
11	3	μ_8	h_5	$[(d_7d_8),(h_5h_2)]$	-
12	3	μ_9	h_7	$[(d_9d_{10}),(h_7u)]$	-
13	3	μ_{10}	h_4	-	-
14	3	μ_{10}	h_8	-	-

4 Conclusions

In this paper, we study stability in matching with couples problems. Stability is essential because stable matchings are robust to rematching by coalitions of agents. In this sense, stable matchings are expected to last and are a good equilibrium prediction.

The presence of complementarities in couples' preferences may prevent the existence of a stable matching (Roth, 1984). As an example of complementarities in couples' preferences, we can think of a couple of married doctors who wish to find jobs in the same city. The couple rejects or accepts job offers for each spouse depending on whether the other spouse can find a job in the same city.

Despite the theoretical impossibility, real-world centralized and decentralized matching markets with couples perform well, suggesting stable matchings are reached. We shed some light on this issue by studying which complementarities are compatible with the possibility of reaching a stable matching through a decentralized matching process.

More precisely, we show that if preferences satisfy restricted complementarity (also known as bilateral substitutability in the matching with contracts literature), then from any arbitrary matching, there is a finite path of matchings such that each matching on the path is obtained by satisfying a blocking coalition from the previous one and the final matching is stable.

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Appendix A

In this appendix, we show that bilateral substitutability is equivalent to restricted complementarity. Bilateral substitutability is a minimal restriction on couples' preferences for the existence of stable matchings (Tello, 2016). However, this condition is formulated on choice functions. Since our objective is to adapt the DPC-Algorithm of Klaus and Klijn (2007), it is necessary to "translate" bilateral substitutability to an equivalent condition on preference orderings. Such a translation is given by restricted complementarity. We first set up a matching with contracts version of the couples problem to prove the equivalence between bilateral substitutability and restricted complementarity. Finally, we show that in this setting: weak substitutability (Hatfield and Kojima, 2008) implies bilateral substitutability, bilateral substitutability implies restricted complementarity, and restricted complementarity implies weak substitutability.

Contracts

A **contract** is an ordered pair $\boldsymbol{(h, d)} \in H \times D$. The ordered pair $\boldsymbol{(u, d)}$ is a **null contract**. All the following definitions in this section are made for couple $c = (d_1, d_2)$ and therefore **we drop all subindexes and function arguments involving it** . The set of all possible contracts with members of couple c is $\boldsymbol{\bar{X}} := (H \cup \{u\}) \times \{d_1, d_2\}$. We denote the set of null contracts involving members of couple c by $\boldsymbol{U} := \big\{(u, d_1), (u, d_2)\big\}$.

The preference relation P over hospital pairs induces a **preference relation** $\tilde{P}$ **over sets of contracts**. Formally, for each (h_1, h_2), $(h_3, h_4) \in \bar{\mathcal{H}}$ we have

$$(h_1, h_2)\, P\, (h_3, h_4) \quad \text{if and only if} \quad \{(h_1, d_1), (h_2, d_2)\}\, \boldsymbol{\tilde{P}}\, \{(h_3, d_1), (h_4, d_2)\}.$$

We define a choice function $\tilde{\mathsf{Ch}}$ as:

$$\boldsymbol{\tilde{\mathsf{Ch}}(X)} := \max_{\tilde{P}}\big\{\{(h, d_1), (h', d_2)\} \subseteq X \cup U \,:\, h, h' \in H \Longrightarrow h \neq h'\big\}, \text{ for each } X \subseteq \bar{X}.$$

We also define a rejection function as:

$$\boldsymbol{\tilde{\mathsf{Rej}}(X)} = X \setminus (\tilde{\mathsf{Ch}}(X) \cup U), \text{ for each } X \subseteq \bar{X}.$$

The rejection function gives for each $X \subseteq \bar{X}$, the set of contracts with hospitals in H that are rejected from X.

It can be easily verified that the choice function $\tilde{\mathsf{Ch}}$ satisfies **consistency** (Alkan, 2002):

$$\tilde{\mathsf{Ch}}(X'') \subseteq (X' \cup U) \subseteq (X'' \cup U) \quad \text{implies} \quad \tilde{\mathsf{Ch}}(X') = \tilde{\mathsf{Ch}}(X'').$$

The relation between Ch and $\tilde{\mathsf{Ch}}$ is stated in Claims 1 and 2 below, but first we need to give some additional definitions.
Let $\mathcal{H} \subseteq \bar{\mathcal{H}}$ we denote,

- the sets of first and second components of the pairs in $\mathcal{H}$ by
  $\boldsymbol{H_1(\mathcal{H})} := \{h : (h, h') \in \mathcal{H}\}$ and $\boldsymbol{H_2(\mathcal{H})} := \{h' : (h, h') \in \mathcal{H}\}$,
- the set of contracts available to c when hospital pairs in $\mathcal{H}$ are available by
  $\boldsymbol{\bar{X}(\mathcal{H})} := \big(H_1(\mathcal{H}) \times \{d_1\}\big) \cup \big(H_2(\mathcal{H}) \times \{d_2\}\big)$.

Let $X \subseteq \bar{X}$ we denote,

- the set of hospitals that have a contract with doctor $d \in \{d_1, d_2\}$ in X by
  $\boldsymbol{H(X, d)} := \{h \in H \cup \{u\} : (h, d) \in X\}$,
- the set of hospitals with contracts in X by
  $\boldsymbol{H(X)} := H(X, d_1) \cup H(X, d_2)$,
- the set of hospital pairs available to c when contracts in X are available by
  $\boldsymbol{\bar{\mathcal{H}}(X)} := \big\{(h, h') \in \bar{\mathcal{H}} \,:\, h \in H(X, d_1) \cup \{u\} \text{ and } h' \in H(X, d_2) \cup \{u\}\big\}$.

Claim 1. For each $X \subseteq \bar{X}$, $\tilde{\mathsf{Ch}}(X) = \{(h, d_1), (h', d_2)\} \implies \mathsf{Ch}(\bar{\mathcal{H}}(X)) = (h, h')$.

Claim 1 follows from the definitions of Ch, $\tilde{\mathsf{Ch}}$, and $\bar{\mathcal{H}}(\cdot)$.

Claim 2. For each complete $\mathcal{H} \subseteq \bar{\mathcal{H}}$, $\mathsf{Ch}(\mathcal{H}) = (h, h') \implies \tilde{\mathsf{Ch}}(\bar{X}(\mathcal{H})) = \{(h, d_1), (h', d_2)\}$.

Claim 2 follows from the definition of a complete set of pairs, and the definitions of Ch, $\tilde{\mathsf{Ch}}$ and $\bar{X}(\cdot)$.

Bilateral substitutability is a property of each couple's choice function $\tilde{\mathsf{Ch}}$. For a couple, it means that if a job offer from hospital h to one of its members is rejected when all other available job offers come from different hospitals, the job offer is still rejected when a new job offer from a different hospital is received.

Bilateral substitutability (Hatfield and Kojima, 2010): there do not exist a set of contracts $X \subseteq \bar{X}$ and contracts $(h, d), (h', d') \in \bar{X}$ such that $h, h' \in H \setminus H(X)$, $(h, d) \notin \tilde{\mathsf{Ch}}\big(X \cup \{(h, d)\}\big)$ and $(h, d) \in \tilde{\mathsf{Ch}}\big(X \cup \{(h, d), (h', d')\}\big)$.

Weak substitutability is a weakening of *bilateral substitutability* and it is not sufficient for the existence of a stable allocation in the contracts problem. Here we present its restriction to the couples problems. Intuitively, it means that the set of job offers rejected by the couple from a set of job offers, where no hospital offers a job to both members of the couple at the same time, expands when the couple receives new job offers from different hospitals.

Weak substitutability (Hatfield and Kojima, 2008): for each $X' \subseteq X'' \subseteq \bar{X}$ such that $\big[(h,d),(h',d') \in X''$ and $h = h' \in H\big]$ imply $[d = d']$, we have $\tilde{\mathsf{Rej}}(X') \subseteq \tilde{\mathsf{Rej}}(X'')$.

Lemma 1. *bilateral substitutability, weak substitutability and restricted complementarity are equivalent.*

Proof:

Weak substitutability $\Longrightarrow$ bilateral substitutability

Follows from the equivalence result in Tello (2016).

Bilateral substitutability $\Longrightarrow$ restricted complementarity

Suppose *restricted complementarity* does not hold. Then, there is a complete $\mathcal{H} \subseteq \bar{\mathcal{H}}$ and h_1, h_2, h_3, h_4 such that $h_1, h_2 \notin \{u, h_3, h_4\}$, $(h_3, h_4) \in \mathcal{H}$, $\mathsf{Ch}(\mathcal{H}) = (h_1, h_2)$, $(h_3, h_4)\, P\, (h_1, h_4)$ and $(h_3, h_4)\, P\, (h_1, u)$.[11] [12]

Let $\boldsymbol{X'} := \{(h_3, d_1), (h_4, d_2), (u, d_1), (u, d_2)\}$ and $\boldsymbol{X''} = \bar{X}(\mathcal{H})$.

Step 1: $(\boldsymbol{h_1}, \boldsymbol{d_1}) \notin \tilde{\mathsf{Ch}}(\boldsymbol{X'} \cup \{(\boldsymbol{h_1}, \boldsymbol{d_1})\})$.

Clearly, $(h_1, d_1), (h_2, d_2) \notin X'$. Moreover, the only pairs in $\bar{\mathcal{H}}(X' \cup \{(h_1, d_1)\})$ involving h_1 are (h_1, h_4) and (h_1, u). By assumption (h_3, h_4) is preferred to both of them. Hence, $(h_1, h_4), (h_1, u) \neq \mathsf{Ch}(\bar{\mathcal{H}}(X' \cup \{(h_1, d_1)\}))$. Therefore by Claim 1, $(h_1, d_1) \notin \tilde{\mathsf{Ch}}(X' \cup \{(h_1, d_1)\})$.

Step 2: $(\boldsymbol{h_1}, \boldsymbol{d_1}) \in \tilde{\mathsf{Ch}}(\boldsymbol{X'} \cup \{(\boldsymbol{h_1}, \boldsymbol{d_1}), (\boldsymbol{h_2}, \boldsymbol{d_2})\})$.

Claim 2 and $\mathsf{Ch}(\mathcal{H}) = (h_1, h_2)$ imply $\tilde{\mathsf{Ch}}(X'') = \{(h_1, d_1), (h_2, d_2)\}$. It also holds that $X' \subseteq X'' \cup U$. Therefore,

$$\tilde{\mathsf{Ch}}(X'') = \{(h_1, d_1), (h_2, d_2)\} \subseteq X' \cup \{(h_1, d_1), (h_2, d_2)\} \subseteq X'' \cup U.$$

Since $\tilde{\mathsf{Ch}}$ satisfies consistency,

$$\tilde{\mathsf{Ch}}(X' \cup \{(h_1, d_1), (h_2, d_2)\}) = \{(h_1, d_1), (h_2, d_2)\}.$$

[11] We are considering the case in which sc1 and sc2 fail. The case in which sc3 and sc4 fail is symmetric.

[12] Strict because $h_1 \neq h_3$.

Step 3: $\boldsymbol{h_1, h_2 \in H \setminus H(X')}$.
This follows from the definition of X' and the assumption that $h_1, h_2 \notin \{u, h_3, h_4\}$.

From Steps 1, 2 and 3, we conclude that *bilateral substitutability* does not hold. $\square$

Restricted complementarity $\Longrightarrow$weak substitutability.
Suppose *weak substitutability* does not hold. Then there are sets $X' \subseteq X'' \subseteq \bar{X}$ such that (i) $\tilde{\mathsf{Rej}}(X') \nsubseteq \tilde{\mathsf{Rej}}(X'')$ and (ii) $[(h,d),(h',d') \in X''$ and $h = h' \in H$ imply $d = d']$.

From (i) there is $(h,d) \in \tilde{\mathsf{Rej}}(X') \setminus \tilde{\mathsf{Rej}}(X'')$. Without loss of generality let $d = d_1$. Let $h_1 = h$. Since $(h,d) \notin \tilde{\mathsf{Rej}}(X'')$ we have, (iii) $\tilde{\mathsf{Ch}}(X'') = \{(h_1,d_1),(h_2,d_2)\}$, for some $h_2 \in H \cup \{u\}$.

Let $\mathcal{H}' = \bar{\mathcal{H}}(X')$ and $\mathcal{H}'' = \bar{\mathcal{H}}(X'')$. By construction, $\mathcal{H}''$ is complete. Moreover, by Claim 1 and (iii) we have

$$\mathsf{Ch}(\mathcal{H}'') = (h_1, h_2) \ \ (\star).$$

Let $h_3, h_4 \in H \cup \{u\}$ be such that:

$$\mathsf{Ch}(\mathcal{H}') = (h_3, h_4) \quad (\star\star).$$

By construction, $(h_3, h_4) \in \mathcal{H}''$.

Step 1. $\boldsymbol{h_1 \notin \{u, h_3\}}$.
By $(\star\star)$ and Claim 2, $(h_3, d_1) \notin \tilde{\mathsf{Rej}}(X')$. Moreover, $(u, d_1) \notin \tilde{\mathsf{Rej}}(X')$ because $\tilde{\mathsf{Rej}}(X')$ only contains contracts with hospitals in H. Since $(h_1, d_1) \in \tilde{\mathsf{Rej}}(X')$, $h_1 \notin \{u, h_3\}$.

Step 2. $\boldsymbol{h_2 \notin \{u, h_4\}}$.
By Step 1, $(h_1, h_2) \neq (h_3, h_4)$. Suppose $h_2 \in \{u, h_4\}$. Then, as $(h_1, d_1) \in X'$ we have

$$\{(h_1,d_1),(h_2,d_2)\} \subseteq X' \cup U \subseteq X'' \cup U.$$

By consistency, $\tilde{\mathsf{Ch}}(X') = \{(h_1,d_1),(h_2,d_2)\}$, but this and Claim 1 contradict $\star\star$.

Step 3. $\boldsymbol{h_1 \neq h_4}$ and $\boldsymbol{h_2 \neq h_3}$.
It follows from $h_1, h_2 \in H$, $(h_1,d_1),(h_2,d_2),(h_3,d_1),(h_4,d_2) \in X''$ and (ii).

Step 4. $\neg$sc1 and $\neg$sc2 hold.
Since $(h_1,d_1),(h_3,d_1),(h_4,d_2) \in X'$, we have $(h_1,h_4),(h_3,h_4),(h_1,u) \in \mathcal{H}'$. Hence, relation $\star\star$ and $h_1 \neq h_3$ imply $(h_3,h_4)\,P\,(h_1,h_4)$ and $(h_3,h_4)\,P\,(h_1,u)$.

From steps 1 to 4, we conclude that P_c does not satisfy *restricted complementarity* for $\mathcal{H}''$. $\square$

Appendix B. The PS-algorithm is well defined

We prove Theorem 1 by showing that the PS-algorithm is well defined for problems where couples' preferences satisfy restricted complementarity, i.e., we prove that given any such problem (P_H, P_C) and any matching μ for (P_H, P_C), the PS-algorithm produces a stable matching for (P_H, P_C) in a finite number of steps.[13]

Proof. We consider the three stages of the PS-algorithm. The first stage clearly is well defined and terminates in a finite number of steps. Also, the matching μ^1 at the end of the first stage does not have blocking hospitals.

The second stage also is well defined and terminates in a finite number of steps: there are only a finite number of couples and hence we only go through the loop a finite number of times. Moreover, the algorithm does not cycle in the loop since hospitals that are dumped are put outside of the room. For the matching μ^2 at the end of the second stage it holds that

- there are no blocking hospitals because (i) matching μ^1 does not have blocking hospitals and (ii) in the second stage all blocking coalitions that may be created in the room are removed by the loop.
- $C \subseteq A$ since the second stage terminates when all couples are in the room.
- for each $d \in D, \mu^2(d) \in A \cup \{u\}$, because (i) when a doctor is put in the room, the hospital he/she is matched to at that moment is put in the room as well and (ii) in the loop the hospitals that are not dumped remain in the room.
- there is no blocking coalition $[c', (h_1', h_2')]$ with $\{h_1', h_2'\} \subseteq A \cup \{u\}$ since in the loop these blocking coalitions are satisfied.

We now proceed to prove that the third stage is well-defined, terminates in a finite number of steps, and that the output of the algorithm μ^3 is a stable matching.

We first prove that the third stage terminates in a finite number of steps. To this end we define a sequence which we prove to be strictly increasing in the number of loops of stage 3 and bounded from above.

For each $c \in C$ and each $(h, h') \in \bar{\mathcal{H}}$ let

$$\boldsymbol{r_c}\,(h, h') := \left|\{(h'', h''') \in \bar{\mathcal{H}} : (h'', h''')\, R_c\, (h, h')\}\right|$$

[13]The proofs in this Appendix follow closely those in Appendix A of Klaus and Klijn (2007).

be the position of (h, h') in the preference list P_c. Denote by μ_k and n_k the matching and the number of hospitals in the room at the beginning of loop k, respectively.

Define the sequence $f_1, f_2, \ldots$ as:

$$f_k := \big(-2 \sum_{c \in C} r_c \left(\mu_k(c)\right)\big) + n_k, \quad k = 1, 2, \ldots$$

At each loop k of the third stage a hospital h' enters the room. Consider two cases.

Case 1. If there is no blocking coalition $[c', (h'_1, h'_2)]$ with $h' \in \{h'_1, h'_2\}$. Then, the matching does not change, i.e., $\mu_{k+1} = \mu_k$. Therefore, $2 \sum_{c \in C} r_c \left(\mu_{k+1}(c)\right) = 2 \sum_{c \in C} r_c \left(\mu_k(c)\right)$. At the same time the number of hospitals increases by one (since h' enters the room and no other hospital leaves it). Hence, $n_{k+1} = n_k + 1$. Hence, $f_{k+1} = f_k + 1$.

Case 2. If there is a blocking coalition $[c', (h'_1, h'_2)]$ with $h' \in \{h'_1, h'_2\}$. Then, from the specific choice we make it follows that at the new matching μ_{k+1} one couple is strictly better off and no other couple changes mates. Hence, $-2 \sum_{c \in C} r_c \left(\mu_{k+1}(c)\right) \geq -2 \sum_{c \in C} r_c \left(\mu_k(c)\right) + 2$. At the same time, hospital h' entered the room and at most two hospitals (which were previously matched to members of the couple in the blocking coalition that was satisfied) exit the room. Therefore, $n_{k+1} \geq n_k - 1$. Summing up the two terms of f_{k+1} we conclude that $f_{k+1} \geq f_k + 1$.

Note that for all $k = 1, 2, \ldots$ the term $-2 \sum_{c \in C} r_c \left(\mu_k(c)\right)$ is bounded from above by $-2|C|$ and the term n_k is bounded from above by $|H|$. So, the sequence $f_1, f_2, \ldots$ is bounded from above by the number $-2|C| + |H|$.

The fact that the sequence $f_1, f_2, \ldots$ is strictly increasing and bounded from above implies that the third stage terminates in a finite number of steps.

It remains to show that the third stage is indeed well defined and that the final matching is stable. It suffices to show that the matching at the beginning of each loop satisfies the following properties:

(i) There is no blocking hospital or blocking couple;

(ii) There is no blocking coalition $[c', (h'_1, h'_2)]$ with $\{h'_1, h'_2\} \subseteq (A \cup \{u\}) \setminus \{h'\}$;

(iii) The Claim holds (which is conditional upon the existence of a blocking coalition $[c', (h'_1, h'_2)]$ with $h' \in \{h'_1, h'_2\} \subseteq A \cup \{u\}$).

Induction Basis: We prove that properties (i)-(iii) hold when the algorithm enters the loop of the third stage for the first time.

(i) and (ii): It follows from the properties of μ^2 that (i) and (ii) hold when the algorithm enters the loop of the third stage for the first time.

(iii): We prove that (iii) holds when the algorithm enters the loop for the first time. Assume that hospital $h' \in H \setminus A$ enters the loop, thus $A = A \cup \{h'\}$. Further, assume that there is a blocking coalition $\left[c', (h'_1, h'_2)\right]$ with $h' \in \{h'_1, h'_2\} \subseteq A \cup \{u\}$. Let d'_1 be h''s most preferred doctor among the ones it would get at these blocking coalitions. Let d'_2 be the partner of d'_1. Without loss of generality we assume $c' = (d'_1, d'_2) \in C$. Let $h^*_1 = \mu(d'_1)$ and $h^*_2 = \mu(d'_2)$.

Suppose to the contrary that (iii) does not hold. Then $\left[c', (h', h^*_2)\right]$, $\left[c', (h', h^*_1)\right]$ and $\left[c', (h', u)\right]$ are not blocking coalitions. Hence, there is a blocking coalition $\left[c', (h', h'_3)\right]$ with $h'_3 \in A \setminus \{h^*_1, h^*_2\}$.

Consider the complete set of pairs $\mathcal{H}''$ depicted in the following table

(1) (h^*_1, h^*_2)	(4) (h', h^*_2)	(7) (u, h^*_2)
(2) (h^*_1, h'_3)	(5) (h', h'_3)	(8) (u, h'_3)
(3) (h^*_1, u)	(6) (h', u)	(9) (u, u)

First, we show that $\mathsf{Ch}_{c'}(\mathcal{H}'') = (h', h'_3)$. By (i), couple c' (weakly) prefers pair (1) to pairs (3), (7), and (9). By (ii), couple c' (strictly) prefers pair (1) to pairs (2) and (8). Since $\left[c', (h', h^*_2)\right]$ and $\left[c', (h', u)\right]$ are not blocking coalitions, couple c' (strictly) prefers (1) to (4) and (6). Finally, since $\left[c', (h', h'_3)\right]$ is a blocking coalition, pair (5) is (strictly) preferred to (1) and therefore, by transitivity, (5) is preferred to all other pairs. This implies (a1) $\mathsf{Ch}_{c'}(\mathcal{H}'') = (h', h'_3)$.

By definition of $\mathcal{H}''$, (a2) $(h^*_1, h^*_2) \in \mathcal{H}''$. Recall $h'_3 \neq u$, h^*_1, h^*_2, and note that since h' just entered the room it must be that $h' \neq u$, h^*_1, h^*_2. Hence, (a3) $h', h'_3 \notin \{u, h^*_1, h^*_2\}$.

From a1, a2, a3 and *restricted complementarity* of P_c follows[14]

$$(h', h^*_2)\, P_{c'}\, (h^*_1, h^*_2) \quad \text{or} \quad (h', u)\, P_{c'}\, (h^*_1, h^*_2).$$

This contradicts the assumption that $\left[c', (h', h^*_2)\right]$ and $\left[c', (h', u)\right]$ are not blocking coalitions. We conclude (iii) holds.

Induction Assumption: Suppose that (i)-(iii) hold for loops 1 up to k of the third stage.

[14]Note (h', h'_3, h^*_1, h^*_2) play the role of (h_1, h_2, h_3, h_4) in the definition of *restricted complementarity*.

Induction Step: Now consider loop $k+1$ (where $k \geq 1$) of the third stage.

Since no agent is forced to accept an unacceptable agent in loop k, (i) is true. Using the arguments for (iii) of the first loop it is easy to check that (iii) is again true for loop $k+1$ if (ii) is also true for loop $k+1$. So, it only remains to prove that (ii) holds for loop $k+1$. It is clear that (ii) holds for loop $k+1$ if there is no blocking coalition $\left[c', (h'_1, h'_2)\right]$ with $h' \in \{h'_1, h'_2\} \subseteq A \cup \{u\}$ for the matching at the end of loop k. We show that in fact this is the case.

Let μ_k and μ_{k+1} be the matchings at the beginning of loops k and $k+1$, respectively.[15] Assume that in loop k blocking coalition $[c', (h', \hat{h})]$ with $c' = (d_1, d_2)$ and

$$(h', \hat{h}) = \mathsf{Ch}_{c'}\Big(\{(h', \mu_k(d'_2)), (h', \mu_k(d'_1)), (h', u)\} \cap \mathcal{B}(c', \mu_k)\Big)$$

is satisfied. In the process of satisfying this blocking coalition, hospitals $\mu_k(d'_1)$ and $\mu_k(d'_2)$ may be dumped. Define h^*_a, h^*_b as follows,

$$h^*_a = \begin{cases} \mu_k(d'_1) & \text{if } \mu_k(d'_1) \text{ is dumped,} \\ u & \text{otherwise;} \end{cases}$$

$$h^*_b = \begin{cases} \mu_k(d'_2) & \text{if } \mu_k(d'_2) \text{ is dumped,} \\ u & \text{otherwise,} \end{cases}$$

then the agents in the room at the beginning of loop $k+1$ are $A \setminus \{h^*_a, h^*_b\}$.
To prove (ii) for loop $k+1$, we have to show that there is no blocking coalition $\left[\bar{c}, (\bar{h}, \tilde{h})\right]$ with $\{\bar{h}, \tilde{h}\} \subseteq (A \setminus \{h^*_a, h^*_b\}) \cup \{u\}$ for μ_{k+1}. Suppose, by contradiction, there is such a blocking coalition. Note that all agents remaining in the room [i.e., all agents in $(A \setminus \{h^*_a, h^*_b\})$] are (weakly) better off at μ_{k+1} compared to μ_k. So, $\left[\bar{c}, (\bar{h}, \tilde{h})\right]$ also blocks μ_k. Hence, if $h' \notin \{\bar{h}, \tilde{h}\}$, then we obtain a contradiction to induction assumption (i) or (ii) for loop k. So, without loss of generality, $(\bar{h}, \tilde{h}) = (h', \tilde{h})$.

If $\bar{c} \neq c'$, then it follows immediately that in loop k hospital h' did not choose its optimal blocking mate; a contradiction. Similarly, if the blocking coalition in question equals $\left[\bar{c}, (\tilde{h}, h')\right]$, then $d'_2 \, P_{h'} d'_1$ and hospital h' did not choose its optimal blocking doctor; a contradiction. Hence, the blocking coalition we consider is of the form $\left[c', (h', \tilde{h})\right]$.

Consider the complete set of pairs $\mathcal{H}''$ depicted in the table below.

First, we show that $\mathsf{Ch}_{c'}(\mathcal{H}'') = (h', \tilde{h})$.

[15] Note that μ_{k+1} is also the matching at the end of loop k.

(1) $(\mu_k(d'_1), \mu_k(d'_2))$	(4) $(h', \mu_k(d'_2))$	(7) $(u, \mu_k(d'_2))$
(2) $(\mu_k(d'_1), \tilde{h})$	(5) $(h', \tilde{h})$	(8) $(u, \tilde{h})$
(3) $(\mu_k(d'_1), u)$	(6) (h', u)	(9) (u, u)

By induction hypothesis (i), couple c' (weakly) prefers pair (1) to pairs (3), (7) and (9). By induction hypothesis (ii), couple c' (strictly) prefers pair (1) to pairs (2) and (8). Now consider the blocking coalition that was satisfied, $\left[c', (h', \hat{h})\right]$. Since $\left[c', (h', \hat{h})\right]$ is a blocking coalition for μ_k, $(h', \hat{h})$ is preferred to (1). By definition, $(h', \hat{h})$ is (weakly) preferred to pairs (4) and (6). Summarizing, the pair $(h', \hat{h})$ is (weakly) preferred to pairs (1),(2),(3),(4),(6),(7),(8) and (9). Lastly, since $\left[c', (h', \tilde{h})\right]$ is a blocking coalition for μ_{k+1} and $\mu_{k+1}(c') = (h', \hat{h})$, $(h', \tilde{h})\, P_{c'}\, (h', \hat{h})$. This implies (b1) $\mathsf{Ch}_{c'}(\mathcal{H}'') = (h', \tilde{h})$. Clearly, (b2) $(\mu_k(d'_1), \mu_k(d'_2)) \in \mathcal{H}''$.

We now show (b3) $h', \tilde{h} \notin \{u,\, \mu_k(d'_1),\, \mu_k(d'_2)\}$. Since h' enters the room in loop k, $h' \notin \{u,\, \mu_k(d'_1),\, \mu_k(d'_2)\}$. Since $\left[c', (h', \tilde{h})\right]$ is a blocking coalition for μ_{k+1}, $\left[c', (h', \tilde{h})\right]$ is also a blocking coalition for μ_k. Hence, by definition of $(h', \hat{h})$, if $\tilde{h} \in \{u,\, \mu_k(d'_1),\, \mu_k(d'_2)\}$, then $(h', \hat{h})\, R_{c'} (h', \tilde{h})$. This contradicts $(h', \tilde{h})\, P_{c'}\, (h', \hat{h})$.

By b1, b2, b3 and *restricted complementarity* of $P_{c'}$, we conclude

$$(\mu_k(d'_1), \tilde{h})\, P_{c'}\, (\mu_k(d'_1), \mu_k(d'_2)) \quad \text{or} \quad (u, \tilde{h})\, P_{c'}\, (\mu_k(d'_1), \mu_k(d'_2)).$$

This contradicts inductive hypothesis (ii). $\square$

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